Numerical methods for fluid and gas dynamics based on regularized equations

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Ecole Normale Supérieure Assia Djebar de Constantine
October 28-31, 2017

A wide variety of **numerical methods for gasdynamic simulations** has been developed up to date. An original class of methods is related to the construction of regularized (quasi-gasdynamic) systems of equations and their subsequent discretization. A detail description of this approach can be found in monographs of B. Chetverushkin and T. Elizarova, etc. One of its advantages is the simplicity of the corresponding parallel implementation.

The law of nondecreasing entropy plays a key role in both physical and mathematical theory of gas dynamics equations, namely,

the Euler equations for an inviscid non-heat-conducting gas and

the Navier–Stokes equations for a viscous heat-conducting gas.

In numerical methods aimed at gasdynamic simulations, the control of total entropy behavior is also an important issue of theory and practice subjected to the growing interest recently.

But the discrete law of nondecreasing entropy is rather frequently overlooked in the development of numerical methods.

This is caused both by the complexity of its derivation and by rather specific requirements imposed on the methods to be constructed.

The similar situation is concerning **the law of non-increasing energy** for the simplified **barotropic systems** that we begin with.

The classical barotropic **Euler** and **Navier-Stokes** systems of equations describing 1D flows of a gas/fluid consist in the following mass and momentum balance equations respectively for the inviscid and viscous cases

$$\partial_t\rho + \partial_x(\rho u) = 0,$$
$$\partial_t(\rho u) + \partial_x(\rho u^2 + p) = \rho F$$

and

$$\partial_t\rho + \partial_x(\rho u) = 0,$$
$$\partial_t(\rho u) + \partial_x(\rho u^2 + p) = \partial_x\Pi + \rho F.$$

Hereafter ∂_t and ∂_x are partial derivatives in $t \geqslant 0$ and $x \in [0, X]$.
The sought functions $\rho > 0$ and u are the density and velocity of a gas, and $p = p(\rho)$ is the pressure; normally $p'(\rho) > 0$.
The Navier-Stokes viscous stress is given by

$$\Pi = \nu\partial_x u.$$

Here F is the density of a given body force and $\nu = \nu(\rho) > 0$ is the scaled viscosity coefficient.

The **regularized** barotropic (quasi-gas dynamics) Euler system of equations in 1D case consists in the following mass and momentum balance equations

$$\partial_t\rho + \partial_x j = 0, \tag{1}$$

$$\partial_t(\rho u) + \partial_x(ju + p) = \partial_x\Pi + \rho_* F. \tag{2}$$

The regularized mass flux is given by

$$j = \rho(u - w), \quad w = \widehat{w} + \frac{\tau}{\rho} u\partial_x(\rho u), \quad \widehat{w} = \frac{\tau}{\rho}\left(\rho u\partial_x u + \partial_x p - \rho F\right) \tag{3}$$

and includes the regularizing velocity w and the relaxation parameter $\tau > 0$. The **regularized viscous stress and the density** are defined by

$$\Pi = \nu\partial_x u + \rho u\widehat{w} + \tau p'(\rho)\partial_x(\rho u), \quad \rho_* = \rho - \tau\partial_x(\rho u). \tag{4}$$

The Euler system ($\mu := 0$ and $\tau := 0$) has the **hyperbolic** type,

the Navier-Stokes system ($\tau := 0$) has the **composite hyperb.-parabolic** type and the regularized system has the **parabolic** type.

Note that it is clear from (3) that more compact form of w is possible:

$$w = \frac{\tau}{\rho}(\partial_x(\rho u^2 + p) - \rho F).$$

We need the function

$$P_0(r) := \int_{r_0}^{r} (r-s)\,\frac{p'(s)}{s}\,ds, \quad r>0,$$

where $r_0 > 0$ is a parameter, and its derivative (known as **the enthalpy**)

$$\mathrm{h}(r) \equiv P_0'(r) = \int_{r_0}^{r} \frac{p'(s)}{s}\,ds = \frac{p(s)}{s}\bigg|_{r_0}^{r} + \int_{r_0}^{r} \frac{p(s)}{s^2}\,ds, \quad r>0.$$

If $p'(r) \geqslant 0$ for $r>0$, then also $P_0(r) \geqslant 0$ for $r>0$ and $P_0(r_0)=0$.
In the adiabatic case, where $p(r) = p_1 r^{\gamma}$ with $\gamma > 1$, one can set $r_0 = 0$, then

$$P_0(r) = \frac{p_1}{\gamma-1}\,r^{\gamma}, \quad \mathrm{h}(r) = \frac{\gamma p_1}{\gamma-1}\,r^{\gamma-1}.$$

In the isothermic case, where $p(r) = p_1 r$, one can set $r_0 = 1$, then

$$P_0(r) = p_1(r\ln r - (r-1)), \quad \mathrm{h}(r) = p_1 \ln r.$$

Assume that the density of the body force has the form

$$F(x,t) = \partial_x \Phi(x),$$

where $\partial_x \Phi$ is the density of the stationary potential body force.

For the regularized barotropic system of equations (33)-(4), the following **energy balance equation** holds

$$\partial_t \left(P_0(\rho) - \rho\Phi + 0.5\rho u^2\right) + \partial_x \left\{j \left(\mathrm{h}(\rho) - \Phi + 0.5u^2\right) - \Pi u\right\}$$

$$+\nu(\partial_x u)^2 + \tau \frac{p'(\rho)}{\rho} \left\{\partial_x(\rho u)\right\}^2 + \tau\rho\{u\partial_x u + \partial_x \mathrm{h}(\rho) - \partial_x\Phi\}^2 = 0. \quad (5)$$

On the left, the 2nd term has the spatial divergent form, and for $p' \geqslant 0$ the 3rd (Navier-Stokes) term and 4th and 5th (relaxation) terms are non-negative.

The last property remains valid for $\nu \geqslant 0$ and $\tau \geqslant 0$.

Notice that, for **the equilibrium solutions** $\rho = \rho_S(x) > 0$ and $u = 0$, the barotropic QGD system (33)-(4) reduces to the equation

$$\partial_x p(\rho_S) = \rho_S \partial_x \Phi \ \text{ on } \ (0, X),$$

or, equivalently, to the equation

$$\mathrm{h}(\rho_S(x)) = \Phi(x) + C \ \text{ on } \ [0, X], \quad (6)$$

the same as for the equilibrium solutions to the compressible barotropic Navier-Stokes equations. Under some assumptions it allows to find ρ_S.

Define on $[0, X]$ an arbitrary nonuniform mesh $\bar{\omega}_h$ with the nodes $0 = x_0 < x_1 < \cdots < x_N = X$ and steps $h_i = x_i - x_{i-1}$.
Let $h_{\max} = \max\limits_{1 \leqslant i \leqslant N} h_i$. Define also an auxiliary (conjugate) mesh ω_h^* with the nodes $x_{i+1/2} = (x_i + x_{i+1})/2$, $0 \leqslant i \leqslant N-1$, and steps $\hat{h}_i = x_{i+1/2} - x_{i-1/2} = (h_i + h_{i+1})/2$.
Let $H(\omega)$ be the space of functions defined on a mesh ω.
For $v \in H(\bar{\omega}_h)$ and $y \in H(\omega_h^*)$, introduce the operators of averaging, shift of the argument and difference quotients

$$[v]_{i+1/2} = 0.5(v_i + v_{i+1}), \quad (v_\pm)_{i+1/2} = v_{i+1/2\pm1/2}, \quad \delta v_{i+1/2} = \frac{v_{i+1} - v_i}{h_{i+1}},$$

$$[y]_i^* = \frac{h_i y_{i-1/2} + h_{i+1} y_{i+1/2}}{2\hat{h}_i}, \quad \delta^* y_i = \frac{y_{i+1/2} - y_{i-1/2}}{\hat{h}_i}.$$

Clearly $[\cdot]$, $(\cdot)_\pm$, δ: $H(\bar{\omega}_h) \to H(\omega_h^*)$ and $[\cdot]^*$, δ^*: $H(\omega_h^*) \to H(\omega_h)$, where $\omega_h = \{x_i;\, 1 \leqslant i \leqslant N-1\}$.

We need the following elementary counterparts of the formula for the derivative of the product of functions

$$\delta(uv) = \delta u \cdot [v] + [u]\delta v, \tag{7}$$

$$\delta^*(y[v]) = \delta^* y \cdot v + [y\delta v]^*, \tag{8}$$

where $u \in H(\bar{\omega}_h)$. To reduce the amount of brackets, we suppose that, for example, $\delta u \cdot [v] = (\delta u)[v]$ (i.e., the sign $\cdot$ cancels the action of the preceding operators from the left).
We also need the formula

$$[y]^* v = \big[y[v]\big]^* - 0.25\delta^*(h_+^2 y\delta v). \tag{9}$$

Let $p' > 0$. We construct the following semi-dicrete mass and momentum balance equations

$$\partial_t \rho + \delta^* j = 0, \tag{10}$$

$$\partial_t (\rho u) + \delta^* (j[u] + [p]) = \delta^* \Pi + [\rho_* F]^* \tag{11}$$

on ω_h. Here we use the following discretizations of the mass flux

$$j = [\rho]_p([u] - w), \tag{12}$$

$$w = \widehat{w} + \frac{\tau}{[\rho]_p}[u]\delta(\rho u), \quad \widehat{w} = \frac{\tau}{[\rho]_p}([\rho]_p[u]\delta u + \delta p - [\rho]_p F), \tag{13}$$

the viscous stress

$$\Pi = \nu \delta u + [\rho]_p [u] \widehat{w} + \tau \widetilde{p'(\rho)} \delta(\rho u), \tag{14}$$

the regularized density and the body force

$$\rho_* = [\rho]_p - \tau \delta(\rho u), \quad F = \delta \Phi, \quad \text{with } \Phi = \Phi(x). \tag{15}$$

The main sought functions $\rho > 0$ and u together with p and Φ are defined on the main mesh $\bar{\omega}_h$ whereas the functions j, w, $\widehat{w}$, Π, ρ_*, τ and ν are defined on the conjugate mesh ω_h^*.
Here together with the simplest averages $[\rho]$ and $[u]$ we apply the nonstandard averages ρ and $p'(\rho)$

$$[\rho]_p = \begin{cases} \dfrac{p(\rho_+) - p(\rho_-)}{\mathrm{h}(\rho_+) - \mathrm{h}(\rho_-)} & \text{for} \quad \rho_+ \neq \rho_- \\ \rho_- & \text{for} \quad \rho_+ = \rho_- \end{cases},$$

$$\widetilde{p'(\rho)} = [\rho]\mathrm{h}(\rho_-; \rho_+).$$

on ω_h^*. Hereafter $g(\alpha; \beta)$ is the divided difference for a function $g \in C^1(0, +\infty)$

$$g(\alpha; \beta) = \frac{g(\beta) - g(\alpha)}{\beta - \alpha} \ \text{for } \alpha \neq \beta, \quad g(\alpha; \alpha) = g'(\alpha), \quad \alpha > 0, \quad \beta > 0.$$

These averages are two-point and symmetric (as the simplest ones) and thus have the $O(h_{\max}^2)$ approximation order for twice differentiable functions ρ and $p = p(r)$.

The formula for $[\rho]_p$ can be rewritten as

$$[\rho]_p = \frac{p(\rho_-;\rho_+)}{\mathrm{h}(\rho_-;\rho_+)}.$$

Note that according to the Cauchy mean value theorem we have

$$\min\{\rho_-,\rho_+\} < [\rho]_p < \max\{\rho_-,\rho_+\} \ \text{ for } \ \rho_- \neq \rho_+.$$

Moreover, in the adiabatic case the defined averages take the form

$$[\rho]_p = \frac{\gamma-1}{\gamma}\frac{\rho_+^\gamma - \rho_-^\gamma}{\rho_+^{\gamma-1} - \rho_-^{\gamma-1}},\ \widetilde{p'(\rho)} = \frac{\gamma p_1}{\gamma-1}\frac{\rho_- + \rho_+}{2}\frac{\rho_+^{\gamma-1} - \rho_-^{\gamma-1}}{\rho_+ - \rho_-} \text{ for } \rho_- \neq \rho_+,$$

and they become the standard ones in the particular case

$$[\rho]_p = [\rho],\ \ \widetilde{p'(\rho)} = 2p_1[\rho] \ \text{ for } \ \gamma = 2.$$

In the isothermic case the defined averages take the form

$$[\rho]_p = \frac{1}{\ln(\rho_-;\rho_+)},\ \ \widetilde{p'(\rho)} = p_1[\rho]\ln(\rho_-;\rho_+) = p_1\frac{[\rho]}{[\rho]_p}.$$

Let us compare the defined averages with the standard or more simple ones. To this end notice that the divided difference has the following integral representation

$$g(\alpha;\beta) = \int_0^1 g'(\alpha(1-s)+\beta s)\,ds. \tag{16}$$

It implies that (in virtue of the Jensen inequality and the definition on convexity), for the convex function g', the following inequalities hold

$$g'\left(\frac{\alpha+\beta}{2}\right) \leqslant g(\alpha;\beta) \leqslant \frac{g'(\alpha)+g'(\beta)}{2};$$

for the concave g', the opposite inequalities hold.
The lower and upper bounds are the midpoint and trapezoid quadrature rules for integral (16) (the inequalities hold also according to the geometrical sense of these rules).

In the adiabatic case, the following integral representations hold

$$[\rho]_p = \int_0^1 \left[\rho_-^{\gamma-1}(1-s) + \rho_+^{\gamma-1}s\right]^{\frac{1}{\gamma-1}} ds,$$

$$\widetilde{p'(\rho)} = \gamma p_1[\rho] \int_0^1 \left[\rho_-(1-s) + \rho_+ s\right]^{\gamma-2} ds.$$

Therefore in virtue of the above inequalities we get

$$[\rho^{\gamma-1}]^{\frac{1}{\gamma-1}} \leqslant [\rho]_p \leqslant [\rho] \ \text{ for } \ 1 < \gamma \leqslant 2,$$

$$[\rho] \leqslant [\rho]_p \leqslant [\rho^{\gamma-1}]^{\frac{1}{\gamma-1}} \ \text{ for } \ \gamma \geqslant 2,$$

together with

$$\gamma p_1[\rho]^{\gamma-1} \leqslant \widetilde{p'(\rho)} \leqslant \gamma p_1[\rho][\rho^{\gamma-2}] \ \text{ for } \ 1 < \gamma \leqslant 2 \ \text{ or } \ \gamma \geqslant 3,$$

$$\gamma p_1[\rho][\rho^{\gamma-2}] \leqslant \widetilde{p'(\rho)} \leqslant \gamma p_1[\rho]^{\gamma-1} \ \text{ for } \ 2 \leqslant \gamma \leqslant 3.$$

Since

$$\ln(\rho_-;\rho_+) = \int_0^1 \frac{1}{\rho_-(1-s)+\rho_+ s}\,ds,$$

in the isothermic case we find

$$\frac{\rho_-\rho_+}{[\rho]} \leqslant [\rho]_p \leqslant [\rho], \quad p_1 \leqslant \widetilde{p'(\rho)} \leqslant p_1\frac{[\rho]^2}{\rho_-\rho_+}.$$

In the practical implementation of the defined averages, to avoid loss of accuracy for $\frac{\rho_+}{\rho_-}\approx 1$, instead of the definitions it is required to apply one or another their approximation, for example, by means of the Taylor expansion in powers of $\frac{\rho_+}{\rho_-}-1$ or, better, a quadrature rule to compute representing integrals like (16).

Note that also the following formula holds

$$[\delta p]_i^* = \delta^*[p]_i = \frac{p_{i+1}-p_{i-1}}{2\hat{h}_i}. \qquad (17)$$

representing the central difference quotient.

Discuss an important property of method (25)-(10). For the equilibrium solutions $\rho = \rho_S(x) > 0$ and $u = 0$ it is reduced to the equations

$$\delta^*\{\tau(\delta p(\rho_S) - [\rho]_p \delta\Phi)\} = 0, \quad \delta^*[p(\rho_S)] = [[\rho]_p \delta\Phi]^* \ \text{ on } \ \omega_h.$$

The first equation implies

$$\delta p(\rho_S) - [\rho]_p \delta\Phi = \frac{C_0}{\tau} \ \text{ on } \ \omega_h^*, \ \text{ with } \ C_0 = \mathrm{const},$$

and in virtue of the above formula $[\delta p]^* = \delta^*[p]$ the second one can be rewritten as

$$[\delta p(\rho_S) - [\rho]_p \delta\Phi]^* = 0,$$

whence $C_0 = 0$. Therefore

$$\delta p(\rho_S) = [\rho]_p \delta\Phi \ \text{ on } \ \omega_h^*.$$

In virtue of the definition of $[\rho]_p$ the last equation takes the form

$$\delta(\mathrm{h}(\rho_S) - \Phi) = 0 \ \text{ on } \ \omega_h^*,$$

thus

$$\mathrm{h}(\rho_S) = \Phi + C \ \text{ on } \ \bar{\omega}_h, \ \text{ with } \ C = \mathrm{const}. \qquad (18)$$

This is the mesh counterpart of the above result for the original system.

Theorem (the discrete energy balance equation)

For the discrete in space method (25)-(10), ***the following energy balance equation*** *holds*

$$\partial_t \left(P_0(\rho) - \rho\Phi + 0.5\rho u^2\right)$$

$$+\delta^* \left\{j\left(\mathrm{h}(\rho) - \Phi + 0.5u_- u_+\right) - \Pi u + B_h\right\}$$

$$+\left[\nu(\delta u)^2 + \tau \mathrm{h}(\rho_-;\rho_+)\left\{\delta(\rho u)\right\}^2 + \tau[\rho]_p\{[u]\delta u + \delta \mathrm{h}(\rho) - \delta\Phi\}^2\right]^* = 0,$$

where $B_h = -0.25h_+^2(\delta p - \rho_*\delta\Phi)\delta u.$
In it all the three summands under the averaging sign $[\cdot]^*$ *on the left are non-negative.*

The last property remains valid for $\nu \geqslant 0$, $\tau \geqslant 0$ and $p' \geqslant 0$ (after replacing in the definition of $\widetilde{p'(\rho)}$ the comparison $\rho_- \neq \rho_+$ by $\mathrm{h}(\rho_-) \neq \mathrm{h}(\rho_+)$).
Clearly the term $u_- u_+$ in the spatially divergent summand is the geometric mean for u^2. Below we show how it appears.

Proof. To illuminate the origin of the above averages $[\rho]_p$ and $\widetilde{p'(\rho)}$ and to demonstrate imbalances appearing for other averages, we consider the method like (25)–(25) but with the following more general expressions for j, w, $\widehat{w}$, Π and ρ_*:

$$j = [\rho]_1([u] - w),$$

$$w = \widehat{w} + \frac{\tau}{[\rho]_2}[u]\delta(\rho u), \quad \widehat{w} = \frac{\tau}{[\rho]_2}([\rho]_2[u]\delta u + \widehat{\delta p} - [\rho]_2 F),$$

$$\Pi = \nu\delta u + [\rho]_3[u]\widehat{w} + \tau\widehat{p'(\rho)}\delta(\rho u),$$

$$\rho_* = [\rho]_4 - \tau\delta(\rho u),$$

where $[\rho]_1$ – $[\rho]_4$ and $\widehat{p'(\rho)}$ are **any averages** for ρ and $p'(\rho)$, as well as $\widehat{\delta p}$ is a discretization of $\partial_x p$ at the nodes of ω_h^* (more general expressions could be also analyzed).

As usual, the derivation of the difference energy balance equation is the difference counterpart of the derivation of the above differential energy balance equation (5).

First we multiply the above mass balance equation $\partial_t \rho + \delta^* j = 0$ by $\mathrm{h}(\rho) - \Phi$. Since

$$\delta^* j \cdot (\mathrm{h}(\rho) - \Phi) = \delta^* \big(j[\mathrm{h}(\rho) - \Phi] \big) - [j \delta(\mathrm{h}(\rho) - \Phi)]^*$$

according to formula (22), we get

$$\partial_t (P_0(\rho) - \rho\Phi) \\ + \delta^* (j[\mathrm{h}(\rho) - \Phi]) - [[\rho]_1 (\delta \mathrm{h}(\rho) - \delta\Phi)\}([u] - w)]^* = 0. \qquad (19)$$

We also multiply the above momentum balance equation $\partial_t(\rho u) + \delta^*(j[u] + [p]) = \delta^*\Pi + [\rho_* F]^*$ by u. We use the formula

$$\partial_t(\rho u) \cdot u = 0.5\partial_t(\rho u^2) + 0.5\partial_t\rho \cdot u^2,$$

apply the mass balance equation and twice the formula $\delta^*(y[v]) = \delta^* y \cdot v + [y\delta v]^*$ and find

$$\partial_t\rho \cdot u^2 = -\delta^* j \cdot u^2 = -\delta^*(j[u^2]) + [j\delta(u^2)]^*,$$
$$\delta^*(j[u]) \cdot u = \delta^*(j[u]^2) - [j[u]\delta u]^*.$$

Then taking into account the elementary formulas

$$[u]^2 = 0.5[u^2] + 0.5u_-u_+, \quad 0.5\delta(u^2) = [u]\delta u \tag{20}$$

together with the above formula $[\delta p] = \delta^*[p]$, we get the equality

$$0.5\partial_t(\rho u^2) + 0.5\delta^*(ju_-u_+) + [\delta p - \rho_*\delta\Phi]^* u - \delta^*\Pi \cdot u = 0.$$

We add it and the above equality (19) containing $P_0(\rho)-\rho\Phi$.
In virtue of the above formulas $[y]^*v=\big[y[v]\big]^*-0.25\delta^*(h_+^2y\delta v)$ and $\delta^*(y[v])=\delta^*y\cdot v+[y\delta v]^*$ we respectively obtain

$$[\delta p-\rho_*\delta\Phi]^*\cdot u=[(\delta p-[\rho]_4\delta\Phi+\tau\delta(\rho u)\cdot\delta\Phi)[u]]^* \\ -0.25\delta^*\left(h_+^2(\delta p-\rho_*\delta\Phi)\delta u\right),$$

$$\delta^*\Pi\cdot u=\delta^*(\Pi[u])-[\Pi\delta u]^*.$$

Therefore we derive

$$\partial_t\left(P_0(\rho)-\rho\Phi+0.5\rho u^2\right)+\delta^*(A+B_h) \\ +[[\rho]_1(\delta\mathrm{h}(\rho)-\delta\Phi)w+\tau[u]\delta(\rho u)\cdot\delta\Phi+\Pi\delta u+D_h]^*=0,$$

with the above defined quantity B_h and

$$A=j\left(\mathrm{h}(\rho)-\Phi+0.5u_-u_+\right)-\Pi u,$$

$$D_h=(\delta p-[\rho]_1\delta\mathrm{h}(\rho))[u]+([\rho]_1-[\rho]_4)[u]\delta\Phi.$$

Clearly the imbalance $D_h=0$ in the case $[\rho]_4=[\rho]_1=[\rho]_p$.

We remind the definitions of w and Π and rewrite the last equality as

$$\partial_t\left(P_0(\rho)-\rho\Phi+0.5\rho u^2\right)+\delta^*(A+B_h)+[\nu(\delta u)^2+\Psi(\rho,u)+D_h]^*=0, \tag{21}$$

where we set

$$\Psi(\rho,u)$$

$$:=[\rho]_1(\delta\mathrm{h}(\rho)-\delta\Phi)w+\tau[u]\delta(\rho u)\cdot\delta\Phi+\{[\rho]_3[u]\widehat{w}+\tau\widehat{p'(\rho)}\delta(\rho u)\}\delta u$$

$$=[\rho]_1(\delta\mathrm{h}(\rho)-\delta\Phi)w+[\rho]_3[u]\delta u\cdot\widehat{w}+\tau\delta\Phi\cdot[u]\delta(\rho u)+\tau\widehat{p'(\rho)}\delta(\rho u)\cdot\delta u.$$

Applying the formulas

$$w=\widehat{w}+\frac{\tau}{[\rho]_2}[u]\delta(\rho u),\quad \widehat{w}=\tau\Big([u]\delta u+\frac{\widehat{\delta p}}{[\rho]_2}-\delta\Phi\Big),$$

we transform $\Psi(\rho,u)$ to the form

we transform $\Psi(\rho,u)$ to the form

$$\Psi(\rho,u)=\{[\rho]_1(\delta\mathrm{h}(\rho)-\delta\Phi)+[\rho]_3[u]\delta u\}\widehat{w}$$

$$+\tau\frac{[\rho]_1}{[\rho]_2}\,\delta\mathrm{h}(\rho)\cdot[u]\delta(\rho u)+\tau\widehat{p'(\rho)}\delta(\rho u)\delta u+\tau\left(1-\frac{[\rho]_1}{[\rho]_2}\right)\delta\Phi\cdot[u]\delta(\rho u)$$

$$=\tau[\rho]_1\left\{\frac{[\rho]_3}{[\rho]_1}[u]\delta u+\delta\mathrm{h}(\rho)-\delta\Phi\right\}\left\{[u]\delta u+\frac{\widehat{\delta p}}{[\rho]_2}-\delta\Phi\right\}$$

$$+\tau\delta(\rho u)\left\{\frac{[\rho]_1}{[\rho]_2}\mathrm{h}(\rho_-;\rho_+)\delta\rho\cdot[u]+\frac{\widehat{p'(\rho)}}{[\rho]}[\rho]\delta u\right\}$$

$$+\tau\left(1-\frac{[\rho]_1}{[\rho]_2}\right)\delta\Phi\cdot[u]\delta(\rho u).$$

Let the equalities

$$[\rho]_3=[\rho]_1=[\rho]_2,\quad \widehat{\delta p}=[\rho]_2\delta\mathrm{h}(\rho),\quad \widehat{p'(\rho)}=[\rho]\mathrm{h}(\rho_-;\rho_+)$$

be valid **as in method** (25)-(10).

Then using also the above formula $\delta\rho \cdot [u] + [\rho]\delta u = \delta(\rho u)$ the quantity $\Psi(\rho, u)$ becomes **the sum of squares**

$$\Psi(\rho, u) = \tau[\rho]_1 \left\{[u]\delta u + \delta \mathrm{h}(\rho) - \delta\Phi\right\}^2 + \tau \mathrm{h}(\rho_-; \rho_+)\{\delta(\rho u)\}^2.$$

Thus we pass from (21) to the energy balance equation in Theorem.

The derived discrete energy balance equation is the mesh counterpart of the differential energy balance equation (5) conserving the non-negativity of the corresponding summands. This is an essential issue and guarantees validity of **the law of non-increasing total energy** (under suitable additional conditions, in the simplest case, under periodicity of the solution in x).

Note that the term $\delta^* B_h$ is the additional summand with respect to the differential case and represents the spatially divergent mesh imbalance. It satisfies the estimate $B_h = O(h_{\max}^2)$ for bounded functions $\rho > 0$ and u of the continuous arguments having the bounded derivatives $\partial_x \rho$, $\partial_x u$ and for bounded τ and $\partial_x \Phi$.

Remark. One can replace $[\rho]_p$ by the simplest one $[\rho]$ in all the equations but together with replacing
$\delta^*[p]$ by $[\widetilde{\delta p}]^*$ with $\widetilde{\delta p} = [\rho]\delta \mathrm{h}(\rho)$ in the momentum balance equation:

$$\partial_t(\rho u) + \delta^*(j[u]) + \big[[\rho]\delta \mathrm{h}(\rho)\big]^* = \delta^*\Pi + [\rho_* F]^*$$

and δp by $\widetilde{\delta p}$ in the formula for $\widehat{w}$:

$$\widehat{w} = \frac{\tau}{[\rho]}\left([\rho][u]\delta u + [\rho]\delta \mathrm{h}(\rho) - [\rho]_p F\right).$$

Then the above discrete energy balance equation (16) remains valid after replacing $[\rho]_p$ by $[\rho]$.

Recall that the regularized **shallow water equations for an uneven bottom** form an important particular case of the above *regularized barotropic Euler system of equations with the potential body force*, namely for $p(\rho) = p_1\rho^2 = 0.5g\rho^2$, $\Phi = -gb$ with $g = \mathrm{const} > 0$,
but now $\rho = \boldsymbol{h}$ has the physical sense of **the depth of water** measured from the bottom mark $b = b(x)$. In this case the above method

$$\partial_t\rho + \delta^* j = 0,$$

$$\partial_t(\rho u) + \delta^*(j[u] + [p]) = \delta^*\Pi + [\rho_*(-g\delta b)]^*$$

is simplified essentially and contain the terms

$$j = [\rho]([u] - w), \quad w = \widehat{w} + \frac{\tau}{[\rho]}[u]\delta(\rho u),$$

$$\widehat{w} = \tau([u]\delta u + g\delta\rho + g\delta b) = \tau\delta\big(g(\rho + b) + 0.5u^2\big),$$

$$\Pi = \nu\delta u + [\rho][u]\widehat{w} + \tau g[\rho]\delta(\rho u);$$

remind that

$$[\rho]_p = [\rho], \quad \widetilde{p'(\rho)} = 2p_1[\rho] \ \text{ for } \ \gamma = 2.$$

Inserting the formula for $\widehat{w}$ into the formula for w gives

$$w = \frac{\tau}{[\rho]}\left(2[\rho][u]\delta u + [u]^2\delta\rho\right) + \tau g\delta(\rho + b).$$

Previously by other authors (O. Bulatov, T. Elizarova) another formula, in an equivalent form:

$$w = \frac{\hat{\tau}}{[\rho]}\,\delta(\rho u^2) + \hat{\tau} g\delta(\rho + b),$$

was used, with $\hat{\tau} = [\tau]$ (where τ was defined on the main mesh). Since

$$\delta(\rho u^2) = 2[\rho][u]\delta u + [u^2]\delta\rho, \quad \text{but} \quad [u]^2 \not\equiv [u^2]$$

the two formulas for w are not identical (though our formula for Π and the formula of that authors are equivalent).
Importantly, in this case the above equality for the equilibrium solution takes the simplest form

$$\rho_S + b = C_1 \quad \text{on} \quad \bar{\omega}_h, \quad \text{with} \quad C_1 = \text{const}.$$

Thus the method is **the well-balanced**.

The above discrete energy balance equation (16) is also simplified and looks like

$$\partial_t \left(g(\rho + b)^2 + 0.5\rho u^2\right) + \delta^* \left\{j\left(g(\rho + b) + 0.5u_- u_+\right) - \Pi u + B_h\right\}$$
$$+ \Big[\nu(\delta u)^2 + \tau g \{\delta(\rho u)\}^2 + \tau[\rho]\{[u]\delta u + g\delta(\rho + b)\}^2\Big]^* = 0;$$

remind that $B_h = -0.25h_+^2(\delta p - \rho_* \delta\Phi)\delta u$.
Remind that actually $\rho = \boldsymbol{h}$ is the water depth and recall that the often appearing above quantity $H = \rho + b$ is **the water level**.
For solving **the inviscid shallow water system**, the viscous and relaxation terms are considered as **artificial regularizers** with μ and τ in the form

$$\mu = \frac{4}{3}\tau[p] \text{ or } \mu = 0, \quad \tau = \alpha\frac{h}{c}, \quad c = \sqrt{g[\boldsymbol{h}]} \text{ on } \omega_h^*; \quad 0 < \alpha < 1,$$

where c is like the velocity of sound and h is the constant spatial step.
We apply **the explicit Euler scheme in time**, and to satisfy a stability condition of the CFL type, choose the time step Δt at the current time as

$$\Delta t = \beta \min_{1 \leqslant i \leqslant N} \frac{h}{\left|[u]_{i-1/2}\right| + c_{i-1/2}}, \quad 0 < \beta < 1.$$

We consider a channel of the length X=25 m with a flat bottom except for the small hump of the parabolic shape in its middle part:

$$b(x) = \begin{cases} 0.2 - 0.05(x-10)^2, & 8\,\mathrm{m} \leqslant x \leqslant 12\,\mathrm{m} \\ 0, & \text{otherwise.} \end{cases}$$

Initially the water level is constant: $H_0(x) \equiv C_H$, and the flow is at rest. The left boundary conditions are $\boldsymbol{h}u|_{x=0} = C_{\boldsymbol{h}u}$ for the discharge together with the open boundary condition for h, and the right boundary conditions are $H|_{x=X} = C_H$ (in general, up to a certain time moment) together with the open boundary condition for u.

This problem may seem simple but only at first glance. There are three types of flows in it: **subcritical**, **transcritical** and **supercritical** depending on values of the flow parameters. Below examples of all of them are considered. Results of computations are presented at $t_{\rm fin}$=200 s (when the flows become stationary for the chosen values of the parameters).

For these flows, the exact discharge at the final time is known: $\boldsymbol{h}u \equiv C_{\boldsymbol{h}u}$.

As a rule, namely the computation of $\boldsymbol{h}u$ causes difficulties.

The results are accurate enough for N listed below; they are comparable with those obtained in other papers.

(a) *Subcritical flow*. This is the simplest type of flows. Here C_H=2 m and $C_{\boldsymbol{h}u}$=4.42 m^2/s are taken. At t_{fin}, the water level is almost flat with only a small cavity above the hump.

We select α=0.9, β=0.2 and μ=0. Fig. 1 shows the water level H and the discharge $\boldsymbol{h}u$ for N=400. In the vicinity of the hump edges, "hubbles" of $\boldsymbol{h}u$ values are observed (damping as N increases) but the absolute error in the discharge $E_{\text{abs}} := \max\limits_{0\leqslant i\leqslant N} |(\boldsymbol{h}u)_i - C_{\boldsymbol{h}u}| \approx 7.375\text{e-}5$ is small.

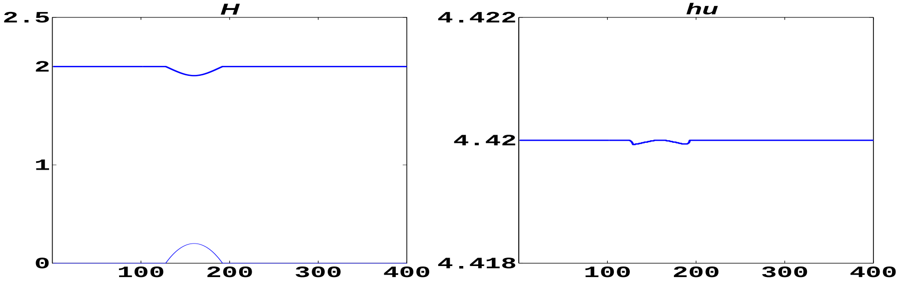


Figure 1: Subcritical flow over the hump: H and $\boldsymbol{h}u$ at $t_{\text{fin}} = 200$ s

(b) *Transcritical flow*. Here C_H=0.66 m and $C_{\boldsymbol{h}u}$=1.53 m^2/s are chosen. For this and the next types of flows, the right boundary condition $H(X,t) = C_H$ is posed only for $t \leqslant 40$ s whereas it is replaced by the open boundary condition for h for $t > 40$ s. The behavior of the stationary water level H exhibits much more sharp change.

We take α=0.9, β=0.1 and μ=0. Fig. 2 shows the level H and the discharge $\boldsymbol{h}u$ at N=400. Once again there are "hubbles" of $\boldsymbol{h}u$ values over the hump edges (damping as N increases) but $E_{\text{abs}} \approx 9.882\text{e-}5$ is small.

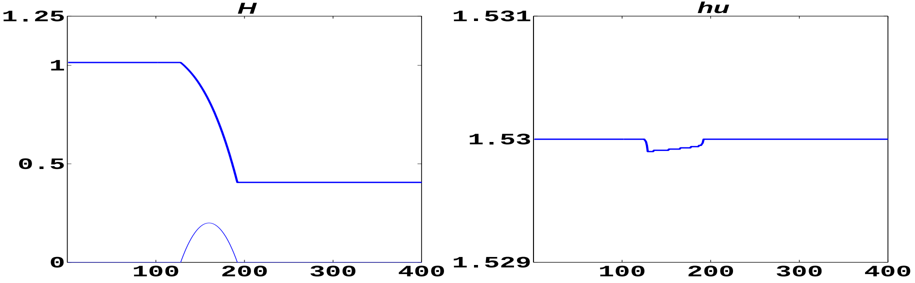


Figure 2: Transcritical flow over the hump: H and $\boldsymbol{h}u$ at $t_{\text{fin}} = 200$ s

(c) *Supercritical flow*. Here C_H=0.33 m and $C_{\boldsymbol{h}u}$=0.18 m^2/s are taken. Now the behavior of the stationary water level H is strongly non–monotone (its graph over the hump has a narrow sharp hollow) and more complicated. We take α=0.8, β=0.1 and N=800. The presence of the Navier–Stokes –type viscosity ($\mu \neq 0$) is essential namely in this case; the stable computations are **impossible** without it. From Fig. 3, we see that $\boldsymbol{h}u$ is now computed worse, see a sharp oscillation near the right edge of the hump. Now $E_{\text{abs}} \approx 0.0266$ is much worse than in the previous two cases.

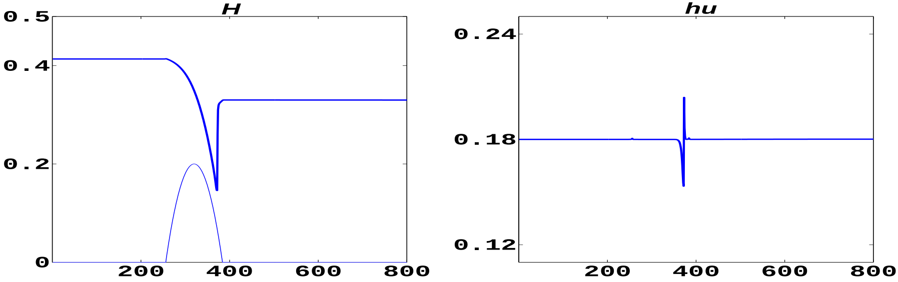
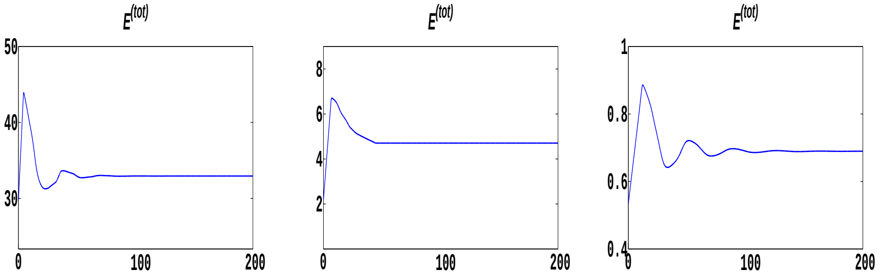


Figure 3: Supercritical flow over the hump: H and $\boldsymbol{h}u$ at $t_{fin} = 200$ s

Finally, in Fig. 4 we present the behavior of the mean total energy

$$E^{(\mathrm{tot})} = \frac{1}{N}(0.5e_0 + \sum_{i=1}^{N-1} e_i + 0.5e_N), \quad e = 0.5g\{(\boldsymbol{h} + b)^2 + \boldsymbol{h}u^2\}$$

in time in all the cases (a), (b) and (c).
To simplify comparison, we take N=100 and the uniform mesh in t.
We observe the stabilization of $E^{(\mathrm{tot})}$ after one or several oscillations (with no any purely numerical oscillations).



(a) (b) (c)

The regularized (quasi-gasdynamic) Euler 1D system of equations consists of (due to B. Chetverushkin and T. Elizarova)
the mass balance equation

$$\partial_t\rho + \partial_x j = 0,$$

the momentum balance equation

$$\partial_t(\rho u) + \partial_x(ju + p) = \partial_x\Pi + \big(\rho - \tau\partial_x(\rho u)\big)F,$$

the total energy balance equation

$$\partial_t E + \partial_x\{(u - w)(E + p)\} = -\partial_x q + \partial_x(\Pi u) + \rho(u - w)F + Q.$$

The unknown functions $\rho > 0$, u and $E = 0.5\rho u^2 + \rho\varepsilon$ are the density, velocity and **total energy of the gas**.
We consider the perfect polytropic gas having **the state equations**

$$p = (\gamma - 1)\rho\varepsilon, \quad \varepsilon = c_V\theta, \quad \text{with} \quad \gamma > 1, \quad c_V > 0.$$

The functions p, ε and $\theta > 0$ are the pressure, internal energy and absolute temperature.

The main equations involve the mass flux with the density

$$j = \rho(u - w),$$

$$w = \widehat{w} + \frac{\tau}{\rho}\, u\partial_x(\rho u), \quad \widehat{w} = \frac{\tau}{\rho}\left(\rho u\partial_x u + \partial_x p - \rho F\right)$$

the viscous stress

$$\Pi = \nu\partial_x u + \rho u\widehat{w} + \tau\left\{u\partial_x p + \gamma p\partial_x u - (\gamma - 1)Q\right\}$$

and the heat flux (with the minus sign)

$$-q = \varkappa\partial_x\theta + \tau\left\{\rho u^2\left(\partial_x\varepsilon - \frac{p}{\rho^2}\,\partial_x\rho\right) - uQ\right\}.$$

Here $\nu = \nu(\rho, \theta)$, $\varkappa = \varkappa(\rho, \theta)$ and $\tau = \tau(\rho, \theta)$ are the viscosity, the heat conductivity and a relaxation parameter (having the dimension of time). Also F and $Q \geqslant 0$ are the body force density and the heat source strength.

Comments. 1. For $\nu = \nu(\rho, \theta) > 0$, $\varkappa = \varkappa(\rho, \theta) > 0$ and $\tau = \tau(\rho, \theta) > 0$ **the regularized Euler (QGD) system** of equations has the *Petrovskii parabolic* type.

2. For $\nu = \nu(\rho, \theta) > 0$, $\varkappa = \varkappa(\rho, \theta) > 0$ and $\tau = 0$, one gets **the compressible Navier-Stokes system** of equations (the viscous heat-conducting gas) having the *composite hyperbolic-parabolic* type.

3. For $\nu = 0$, $\varkappa = 0$ and $\tau = 0$, one gets **the Euler system of equations** (the inviscid non-heat-conducting gas) having the *hyperbolic* type.
All the three cases are covered in this report.

4. The specific coefficients are exploited to solve the Euler system of equations numerically:

$$\nu = \tau p \mathrm{Sc},\ \varkappa = \tau p \frac{\gamma \mathrm{Sc}}{c_V \mathrm{Pr}},\ \tau = \alpha \frac{h}{c},$$

where $\mathrm{Sc} > 0$ and $\mathrm{Pr} > 0$ are the Schmidt and Prandtl numbers, $c = \sqrt{\gamma(\gamma - 1)\varepsilon}$ is the speed of sound while h is the mesh size and $0 < \alpha < 1$ is a parameter.

The internal energy balance equation

$$\partial_t(\rho\varepsilon) + \partial_x(j\varepsilon) = -\partial_x q + \Pi\partial_x u - p\partial_x(u - w) + w\partial_x p - \rho\widehat{w}F + Q.$$

The entropy of the perfect polytropic gas is defined by

$$s = -k\ln\rho + c_V\ln\theta, \quad \text{with} \quad k = (\gamma - 1)c_V.$$

One can derive ***the entropy balance equation***

$$\partial_t(\rho s) + \partial_x(js) = \partial_x\left(-\frac{q}{\theta}\right) + \frac{\varkappa(\partial_x\theta)^2}{\theta^2} + \frac{\nu(\partial_x u)^2}{\theta} + \frac{\rho\widehat{w}^2}{\tau\theta}$$
$$+ \frac{\tau k}{\rho}\{\partial_x(\rho u)\}^2 + \frac{\tau c_V\rho}{\theta^2}\left\{(\gamma - 1)\theta\partial_x u + u\partial_x\theta - \frac{Q}{2c_V\rho}\right\}^2 + \frac{Q}{\theta}\left(1 - \frac{\tau Q}{4\rho\varepsilon}\right).$$

The sum of all terms on the right, except the first divergent one, represents **the entropy production**. Its first five terms are always nonnegative; the last term is nonnegative if $\tau Q \leqslant 4\rho\varepsilon$.

We use a uniform mesh $\bar{\omega}_h$ on $[0, X]$ with the nodes $x_i = ih$, $0 \leqslant i \leqslant N$
and the step $h = X/N$
and the auxiliary mesh ω_h^* with the nodes $x_i = (i + 1/2)h$, $0 \leqslant i \leqslant N - 1$.
Define the mesh-averaging, argument-shift and difference quotient operators

$$[v]_{i+1/2} = 0.5(v_i + v_{i+1}), \quad (v_\pm)_{i+1/2} = v_{i+1/2\pm1/2}, \quad \delta v_{i+1/2} = \frac{v_{i+1} - v_i}{h},$$
$$[y]_i^* = 0.5(y_{i-1/2} + y_{i+1/2}), \quad \delta^* y_i = \frac{y_{i+1/2} - y_{i-1/2}}{h},$$

where $v \in H(\bar{\omega}_h)$ and $y \in H(\omega_h^*)$.
Thus $[\cdot], (\cdot)_\pm, \delta$: $H(\bar{\omega}_h) \to H(\omega_h^*)$
while $[\cdot]^*, \delta^*$: $H(\omega_h^*) \to H(\omega_h)$ with $\omega_h = \{x_i;\, 1 \leqslant i \leqslant N - 1\}$.

Several different counterparts of the product rule are valid:

$$\delta(uv) = \delta u \cdot [v] + [u]\delta v,$$

$$\delta^*(y[v]) = \delta^* y \cdot v + [y\delta v]^*,$$

$$\delta^*\left([u][v] - 0.25h^2\delta u \cdot \delta v\right) = \delta^*[u] \cdot v + u\delta^*[v],$$

where $u \in H(\bar{\omega}_h)$. Hereafter, for example, $\delta u \cdot [v] = (\delta u)[v]$ (i.e., the sign $\cdot$ terminates the action of the preceding operators from the left).
Additional useful formulas

$$[y]^* v = \big[y[v]\big]^* - 0.25h^2\delta^*(y\delta v),$$

$$[uv] = [u][v] + 0.25h^2\delta u \cdot \delta v.$$

The semidiscrete mass, momentum and total energy balance equations are

$$\partial_t \rho + \delta^* j = 0,$$

$$\partial_t (\rho u) + \delta^* (j[u] + [p]) = \delta^* \Pi,$$

$$\partial_t E + \delta^* \left\{ ([u] - w)([E]_2 + [p]) - 0.25 h^2 \delta u \cdot \delta p \right\} = \delta^* (-q + \Pi[u]) + [Q]^*$$

on ω_h (for $F = 0$), with the pressure, total energy and internal energy

$$p = (\gamma - 1)\rho\varepsilon, \quad E = 0.5\rho u^2 + \rho\varepsilon, \quad \varepsilon = c_V \theta.$$

The discretizations of the mass flux density

$$j = [\rho]_{\ln}([u] - w), \quad w = \widehat{w} + \frac{\tau}{[\rho]}[u]\delta(\rho u), \quad \widehat{w} = \frac{\tau}{[\rho]}([\rho][u]\delta u + \delta p)$$

the viscous stress and the heat flux (with the minus sign)

$$\Pi = \nu \delta u + [\rho][u]\widehat{w} + \tau \left\{ [u]\delta p + \gamma [p]_1 \delta u - (\gamma - 1) Q \right\},$$

$$-q = \varkappa \delta\theta + \tau \left\{ [\rho][u]^2 \left(\delta\varepsilon - \frac{[p]_1}{[\rho]^2} \delta\rho \right) - [u]Q \right\}.$$

The basic unknown functions $\rho > 0$, u, E together with p, ε, $\theta > 0$ are defined on $\bar{\omega}_h$ while j, w, $\widehat{w}$, Π, q, τ, ν, $\varkappa$, Q are defined on ω_h^*.
The standard arithmetic averages $[\rho]$, $[u]$, $[p]$ are combined with the nonstandard ones for ρ, p, E, ε: $[\rho]_{\ln} = \frac{1}{\ln(\rho_-; \rho_+)}$; remind that $\ln(\alpha; \beta)$ is the divided difference for the logarithmic function

$$\ln(\alpha;\beta) = \frac{\ln\beta - \ln\alpha}{\beta - \alpha} \text{ for } \alpha \neq \beta, \ \ \ln(\alpha;\alpha) = \frac{1}{\alpha}, \ \alpha > 0, \ \beta > 0$$

$[p]_1 = (\gamma - 1)[\rho][\varepsilon]$

$[E]_2 = 0.5[\rho]_{\ln} u_- u_+ + [\rho]_{\ln}[\varepsilon]_3$, where $u_- u_+$ is the geometric mean for u^2

$[\varepsilon]_3 = \ln\left(\frac{1}{\varepsilon}_-; \frac{1}{\varepsilon}_+\right) = \varepsilon_-\varepsilon_+ \ln(\varepsilon_-; \varepsilon_+)$,

For $\beta/\alpha \approx 1$, to ensure the computational stability, the formulas should be approximated, for example, by the Simpson formula

$$\ln(\alpha;\beta) = \int_0^1 \frac{1}{(1-\alpha)s + \beta s}\, ds \approx \frac{1}{6\alpha} + \frac{4}{3(\alpha+\beta)} + \frac{1}{6\beta}.$$

The entropy balance equation is based on the mass and internal energy balance ones. We multiply the momentum equation by u. We exploit

$$\partial_t(\rho u)\cdot u = 0.5\partial_t(\rho u^2) + 0.5\partial_t\rho\cdot u^2,$$

the mass balance equation and the second difference product rule and get

$$\partial_t\rho\cdot u^2 = -\delta^* j\cdot u^2 = -\delta^*(j[u^2]) + [j\delta(u^2)]^*,$$
$$\delta^*(j[u])\cdot u = \delta^*(j[u]^2) - [j[u]\delta u]^*.$$

Taking into account the equalities

$$[u]^2 = 0.5[u^2] + 0.5u_-u_+, \quad 0.5\delta(u^2) = [u]\delta u,$$

we derive **the kinetic energy balance equation**

$$0.5\partial_t(\rho u^2) + 0.5\delta^*(ju_-u_+) + \delta^*[p]\cdot u = \delta^*\Pi\cdot u.$$

Subtracting the last equation from the total energy balance equation and by the difference product rules get

$$\delta^*\left([u][p] - 0.25h^2\delta u \cdot \delta p\right) = \delta^*[u] \cdot p + \delta^*[p] \cdot u,$$
$$\delta^*(w[p]) = \delta^* w \cdot p + [w\delta p]^*, \quad \delta^*(\Pi[u]) = \delta^*\Pi \cdot u + [\Pi\delta u]^*.$$

Thus the following **internal energy balance equation** holds:

$$\partial_t(\rho\varepsilon) + \delta^*(j[\varepsilon]_3) = -\delta^* q + [\Pi\delta u]^* - p\delta^*([u] - w) + [w\delta p]^* + [Q]^*.$$

At this stage, the form of $u_- u_+$ and the additional summand $0.25h^2\delta u \cdot \delta p$ in the term $[E]_2$ have already played their role: the kinetic and internal energy balance equations contain no mesh imbalances.

But the particular form of $[\rho]_{\ln},\ [p]_1,\ [\varepsilon]_3$ have not yet been used and they could be arbitrary.

Theorem (the discrete entropy balance equation)

For the above spatially discrete method, the following entropy balance equation holds:

$$\partial_t(\rho s) + \delta^*(j[s])$$

$$= \delta^*\left(-q\left[\frac{1}{\theta}\right] + B_h\right) + \left[\frac{\varkappa(\delta\theta)^2}{\theta_-\theta_+} + \frac{\nu[\theta](\delta u)^2}{\theta_-\theta_+} + \frac{[\rho][\theta]}{\tau\theta_-\theta_+}\widehat{w}^2\right.$$

$$+\frac{\tau k[\theta]^2}{[\rho]\theta_-\theta_+}\{\delta(\rho u)\}^2 + \frac{\tau c_V[\rho]}{\theta_-\theta_+}\left\{[u]\delta\theta + (\gamma-1)[\theta]\delta u - \frac{Q}{2c_V[\rho]}\right\}^2$$

$$\left.+\frac{[\theta]}{\theta_-\theta_+}Q\left(1 - \frac{\tau Q}{4[\rho][\varepsilon]}\right)\right]^*.$$

Here, the first five terms in the entropy production (under the sign $[\cdot]^*$) are always nonnegative while the last term is nonnegative if $\tau Q \leqslant 4[\rho][\varepsilon]$.

Let us turn to derivation of the entropy balance equation. According to the definition of the entropy

$$\partial_t(\rho s) = \partial_t\rho\cdot s + \rho\left(-\frac{k}{\rho}\,\partial_t\rho + \frac{c_V}{\theta}\,\partial_t\theta\right) = \partial_t\rho\cdot s - (k+c_V)\partial_t\rho + \partial_t(\rho\varepsilon)\cdot\frac{1}{\theta}.$$

Combining the mass balance equation and $\delta^*(j[s]) = \delta^* j\cdot s + [j\delta s]^*$ yields

$$\partial_t(\rho s) + \delta^*(j[s]) = [j\delta s]^* + \partial_t(\rho\varepsilon)\cdot\frac{1}{\theta} + (k+c_V)\delta^* j.$$

Since

$$\delta^*\left(j[\varepsilon]_3\left[\frac{1}{\theta}\right]\right) = \left[j[\varepsilon]_3\delta\frac{1}{\theta}\right]^* + \delta^*(j[\varepsilon]_3)\cdot\frac{1}{\theta},$$

we can write

$$\partial_t(\rho s) + \delta^*(j[s]) = \left[j\left(\delta s + [\varepsilon]_3\delta\frac{1}{\theta}\right)\right]^* + \{\partial_t(\rho\varepsilon) + \delta^*(j[\varepsilon]_3)\}\,\frac{1}{\theta}$$
$$+\delta^*\left\{j\left(k + c_V - [\varepsilon]_3\left[\frac{1}{\theta}\right]\right)\right\}.$$

Applying the equalities

$$\delta s = -k\delta\ln\rho + c_V\delta\ln\theta = -k\ln(\rho_-;\rho_+)\delta\rho + c_V\ln(\varepsilon_-;\varepsilon_+)\delta\varepsilon$$

and elementary formulas

$$\delta\frac{1}{\theta} = -\frac{\delta\theta}{\theta_-\theta_+}, \quad \left[\frac{1}{\theta}\right] = \frac{[\theta]}{\theta_-\theta_+},$$

we transform the first term

$$j\left(\delta s + [\varepsilon]_3\delta\frac{1}{\theta}\right) = -[\rho]_{\ln}([u]-w)k\ln(\rho_-;\rho_+)\delta\rho$$
$$+j\left(c_V^2\ln(\varepsilon_-;\varepsilon_+) - [\varepsilon]_3\frac{1}{\theta_-\theta_+}\right)\delta\theta = -k([u]-w)\delta\rho,$$

Namely the validity of the last so simple equality is ensured by the above choice of $[\rho]_{\ln}$ and $[\varepsilon]_3$.

Let us transform the second term. By virtue of the internal energy balance equation and the previous difference formulas

$$\{\partial_t(\rho\varepsilon) + \delta^*(j[\varepsilon]_3)\}\,\frac{1}{\theta}$$
$$= -\delta^*([u]-w)\cdot k\rho - \delta^* q\cdot\frac{1}{\theta} + [\Pi\delta u + w\delta p + Q]^*\frac{1}{\theta}$$
$$= \left[k([u]-w)\delta\rho + q\delta\frac{1}{\theta} + (\Pi\delta u + w\delta p + Q)\left[\frac{1}{\theta}\right]\right]^*$$
$$-\delta^*\left\{k([u]-w)[\rho] + q\left[\frac{1}{\theta}\right] + 0.25h^2(\Pi\delta u + w\delta p + Q)\delta\frac{1}{\theta}\right\}.$$

Consequently from above formulas and definitions of q and Π we derive

$$\partial_t(\rho s)+\delta^*(j[s]) = \delta^*\left(-q\left[\frac{1}{\theta}\right]+B_h\right)+\left[\frac{\varkappa(\delta\theta)^2}{\theta_-\theta_+}+\frac{\nu[\theta](\delta u)^2}{\theta_-\theta_+}+\frac{A}{\theta_-\theta_+}\right]^*,$$

where the term $[\dots]^*$ represents **the entropy production**, with

$$A := \tau\left\{[\rho][u]^2\left(\delta\varepsilon-\frac{[p]_1}{[\rho]^2}\,\delta\rho\right)-[u]Q\right\}\delta\theta$$

$$+\left\{\big([\rho][u]\widehat{w}+\tau([u]\delta p+\gamma[p]_1\delta u-(\gamma-1)Q)\big)\delta u+w\delta p+Q\right\}[\theta],$$

$$B_h := kj\left(1-\frac{[\rho]}{[\rho]_{\ln}}\right)+c_V j\left(1-[\varepsilon]_3\left[\frac{1}{\varepsilon}\right]\right)-0.25h^2(\Pi\delta u+w\delta p+Q)\delta\frac{1}{\theta}.$$

Let us show that $A \geqslant 0$.
Let us divide A into the sum of terms containing multipliers $\widehat{w}$ и w, other terms without the multiplier Q and the terms involving Q:

$$A = [\theta]A' + \tau A'' - \tau \left\{[u]\delta\theta + (\gamma - 1)[\theta]\delta u\right\} Q + [\theta]Q.$$

The terms of A' are rearranged as follows:

$$A' \equiv [\rho][u]\widehat{w}\delta u + w\delta p = \widehat{w}[\rho][u]\delta u + \left(\widehat{w} + \frac{\tau}{[\rho]}[u]\delta(\rho u)\right)\delta p$$

$$= \widehat{w}([\rho][u]\delta u + \delta p) + \frac{\tau}{[\rho]}[u]\delta(\rho u) \cdot k(\delta\rho \cdot [\theta] + [\rho]\delta\theta)$$

$$= \frac{[\rho]}{\tau}\widehat{w}^2 + \frac{\tau k[\theta]}{[\rho]}\delta(\rho u) \cdot \delta\rho \cdot [u] + \tau k[u]([\rho]\delta u + [u]\delta\rho)\delta\theta,$$

where the simplest difference product rule has been used twice.

Next, the terms A'' are regrouped so that

$$A'' \equiv [\rho][u]^2 \left(\delta\varepsilon - \frac{[p]_1}{[\rho]^2}\,\delta\rho\right)\delta\theta + \big([u]\delta p + \gamma[p]_1\delta u\big)\delta u \cdot [\theta]$$

$$= c_V[\rho]([u]\delta\theta)^2 - k[\theta][u]^2\delta\rho\cdot\delta\theta$$

$$+k[u](\delta\rho\cdot[\theta] + [\rho]\delta\theta])\delta u\cdot[\theta] + \gamma k[\rho]([\theta]\delta u)^2$$

$$= c_V[\rho]([u]\delta\theta)^2 - k[\theta][u]^2\delta\rho\cdot\delta\theta$$

$$+k[\theta]^2(\delta\rho\cdot[u] + [\rho]\delta u)\delta u + k[\rho][u]\delta u\cdot[\theta]\delta\theta$$

$$+c_V(\gamma-1)^2[\rho]([\theta]\delta u)^2$$

since $\gamma k = k + c_V(\gamma-1)^2$.
In the last transformations an important role has been played by the form of $[p]_1$ in the expressions for the viscous stress and the heat flux.

Thus the following terms are rearranged into a sum of squares:

$$[\theta]A' + A'' = \frac{[\rho][\theta]}{\tau}\,\widehat{w}^2 + \frac{\tau k[\theta]^2}{[\rho]}\,\delta(\rho u)\cdot\big(\delta\rho\cdot[u] + [\rho]\delta u\big)$$

$$+\tau\left\{c_V[\rho]([u]\delta\theta)^2 + 2k[\rho][u]\delta u\cdot[\theta]\delta\theta + c_V[\rho]((\gamma-1)[\theta]\delta u)^2\right\}$$

$$= \frac{[\rho][\theta]}{\tau}\,\widehat{w}^2 + \frac{\tau k[\theta]^2}{[\rho]}\,\{\delta(\rho u)\}^2 + \tau c_V[\rho]\big\{[u]\delta\theta + (\gamma-1)[\theta]\delta u\big\}^2.$$

Finally, taking into account the terms with Q, we pass to the discrete entropy balance equation.

Comments. 1. As in the differential case, the entropy production remains nonnegative for $\nu(\rho,\varepsilon) \geqslant 0$, $\varkappa(\rho,\varepsilon) \geqslant 0$ and $\tau(\rho,\varepsilon) \geqslant 0$ since the third summand under the sign $[\cdot]^*$ can be rewritten in the form

$$\frac{\tau[\theta]}{[\rho]\theta_-\theta_+}\left([\rho][u]\delta u + \delta p\right)^2.$$

2. The additional term $\delta^* B_h$ (absent in the differential case) is a divergent difference imbalance. Clearly $B_h = O(h^2)$ for continuous functions $\rho > 0$, u and $\theta > 0$ with bounded derivatives $\partial_x^2 \rho$, $\partial_x u$ and $\partial_x^2 \theta$ and for bounded τ, ν and Q.

3. The first two terms in B_h appear since $[\rho]_{\ln}$ and $[\varepsilon]_3$ are used instead of the simplest $[\rho]$ and $[\varepsilon]$. Otherwise, they disappear but instead such additions to the entropy production arise that one cannot any more guarantee its nonnegativity.

4. The results are generalized for **an arbitrary non-uniform mesh** $\bar{\omega}_h$ on $[0, X]$.

For comparison, we consider also the more standard discretization of the regularized system (for $F = 0$) (T. Elizarova and E. Shilnikov).
Its basic equations are (the differences are marked in green)

$$\partial_t \rho + \delta^* j = 0,$$

$$\partial_t(\rho u) + \delta^*(j[u] + [p]) = \delta^* \Pi,$$

$$\partial_t E + \delta^* \{([u] - w)([E] + [p])\} = \delta^*(-q + \Pi[u]) + Q$$

on ω_h, with p, E and ε related by the previous standard relations. In contrast to above, Q is defined on $\bar{\omega}_h$.
The additional relations are

$$j = [\rho]([u] - w), \quad [E] = 0.5[\rho]_0[u]^2 + [\rho\varepsilon],$$

$$w = \frac{\tau}{[\rho]}\,\delta(\rho u^2 + p), \quad \hat{w} = \frac{[\tau]}{[\rho]}\,([\rho][u]\delta u + \delta p),$$

$$\Pi = \nu\delta u + [\rho][u]\hat{w} + \tau([u]\delta p + \gamma[p]\delta u - (\gamma - 1)[Q]),$$

$$-q = \varkappa\delta\theta + \tau\left\{[\rho][u]^2\left(\delta\varepsilon + [p]\delta\frac{1}{\rho}\right) - [u][Q]\right\}$$

on ω^*

Theorem (the standard discrete entropy balance equation)

For the standard spatially discrete method, the following entropy balance equation holds:

$$\partial_t(\rho s) + \delta^*(j[s]) = \delta^*\left\{-q\left[\frac{1}{\theta}\right] - \alpha_h\left[\frac{1}{\theta}\right] + \beta_h\right\}$$

$$+\left[\frac{\varkappa(\delta\theta)^2}{\theta_-\theta_+} + \frac{\nu[\theta](\delta u)^2}{\theta_-\theta_+} + \frac{[\rho][\theta]}{\tau\theta_-\theta_+}\hat{w}^2 + \frac{\tau k[\theta]^2}{[\rho]\theta_-\theta_+}\{\delta(\rho u)\}^2\right.$$

$$+\frac{\tau c_V[\rho]}{\theta_-\theta_+}\left\{[u]\delta\theta + (\gamma-1)[\theta]\delta u - \frac{[Q]}{2c_V[\rho]}\right\}^2$$

$$\left.+\frac{[\theta]}{\theta_-\theta_+}[Q]\left(1 - \frac{\tau[Q]}{4[\rho][\varepsilon]}\right) + \xi_h + \frac{\tau\zeta_h}{\theta_-\theta_+}\right]^*,$$

where α_h, β_h, ξ_h and ζ_h are imbalance terms of $indefinite\ signs$.

The precise expressions for the imbalance terms are rather complicated:

$$\alpha_h = 0.25h_+^2 \left\{0.5j(\delta u)^2 + \delta u \cdot \delta p\right\},$$

$$\beta_h = c_V j \left(1 - \frac{[\rho\varepsilon]}{[\rho]}\left[\frac{1}{\varepsilon}\right]\right) - 0.25h_+^2 \left\{(\Pi\delta u + w\delta)\delta\frac{1}{\theta} + \delta\frac{Q}{\theta}\right\},$$

$$\xi_h = kj\left(1 - \frac{[\rho]}{[\rho]_{\ln}}\right)\delta\rho + j\left(\frac{[\rho\varepsilon]}{[\rho]} - [\varepsilon]_3\right)\delta\frac{1}{\theta} + \alpha_h\delta\frac{1}{\theta} + 0.25h_+^2\delta Q \cdot \delta\frac{1}{\theta},$$

$$\zeta_h = (\gamma - 1)[u]^2\left([\varepsilon] - \frac{[\rho][\rho\varepsilon]}{\rho_-\rho_+}\right)\delta\rho \cdot \delta\theta + \gamma(\gamma - 1)([\rho\varepsilon] - [\rho][\varepsilon])(\delta u)^2[\theta]$$

$$+\frac{1}{[\rho]}([\rho u] - [\rho][u])\delta u \cdot \delta p \cdot [\theta].$$

The following error estimate holds:

$$|\alpha_h| + |\beta_h| + |\xi_h| + |\zeta_h| = O(h_{\max}^2)$$

for continuous $\rho > 0$, u, $\theta > 0$ and Q with bounded derivatives $\partial_x\rho$, $\partial_x u$, $\partial_x\theta$ and $\partial_x Q$.

However, the methods are mainly used to compute *discontinuous solutions*.

Test 1: The Sod problem

In all the tests, we apply the method to solve the Euler system of equations. We exploit the simplest explicit approximation in time (excepting test 3, where we take the second order explicit midpoint approximation).

Test 1. A version of the Sod problem. The resulting flow involves all the characteristic features of supersonic flows: sonic points at the boundaries of a rarefaction wave, a contact discontinuity and a shock wave. Here the computational domain is $[-0.5, 0.5]$, $\gamma = 1.4$, and the initial data are

$$\rho_0(x) = \begin{cases} 1, & x \leqslant 0 \\ 0.125, & x > 0 \end{cases}, \quad p_0(x) = \begin{cases} 1, & x \leqslant 0 \\ 0.1, & x > 0 \end{cases}, \quad u_0(x) = \begin{cases} 0.75, & x \leqslant 0 \\ 0, & x > 0 \end{cases}$$

We take $t_{fin} = 0.2$, $N = 400$, $\alpha = 0.4$ and the time step

$$\Delta t = \beta \min_i \frac{h}{|u_i| + c_i}$$

with $\beta = 0.2$.

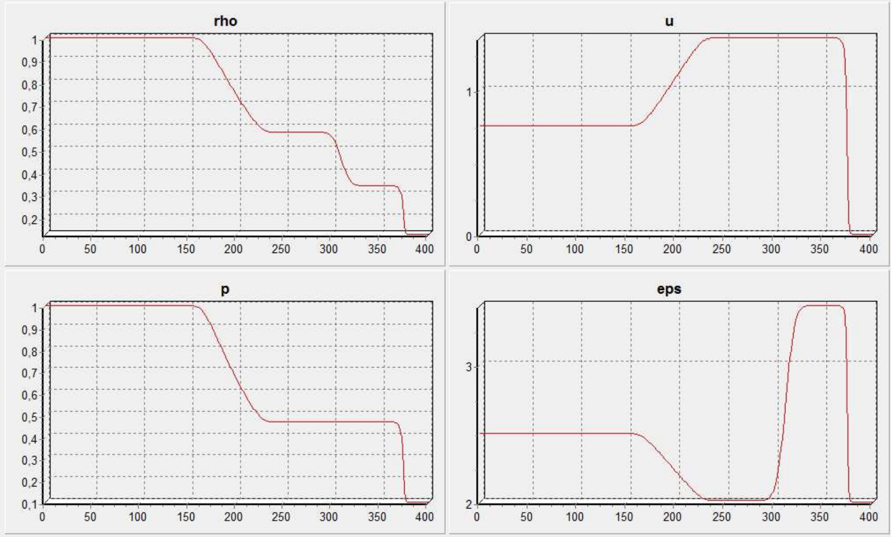


Figure 5: The Sod problem

Test 2: Two opposite rarefaction waves

The flow represents two rarefaction waves that propagate away from the center of the domain. The difficulty in the numerical solution of this problem is that the gas density and pressure at the center (between the diverging flows) are very low, while the internal energy ε does not tend to zero.

P.R. Woodward and P. Colella: "It seems that there are no difference schemes in the Eulerian variables that describe the behavior of the internal energy in this problem with high accuracy."

Here the computational domain is $[-0.5, 0.5]$, $\gamma = 1.4$ and the initial data are

$$\rho_0(x) = 1,\ p_0(x) = 0.4,\ u_0(x) = \begin{cases} -2, & x \leqslant 0 \\ 2, & x > 0 \end{cases}.$$

We take $t_{fin} = 0.15$ together with $N = 500$, $\alpha = 0.018$ and $\beta = 0.09$.

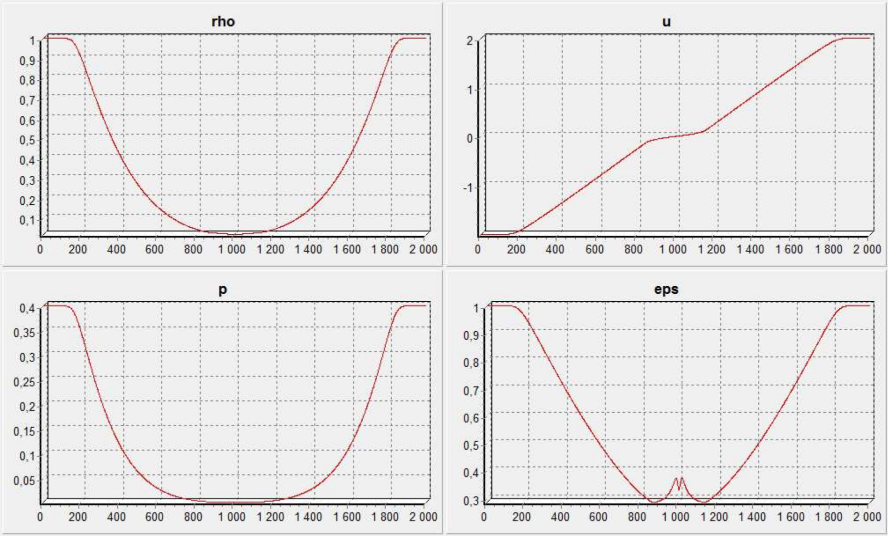


Figure 6: Two opposite rarefaction waves (results for the standard QGD scheme, N=2000)

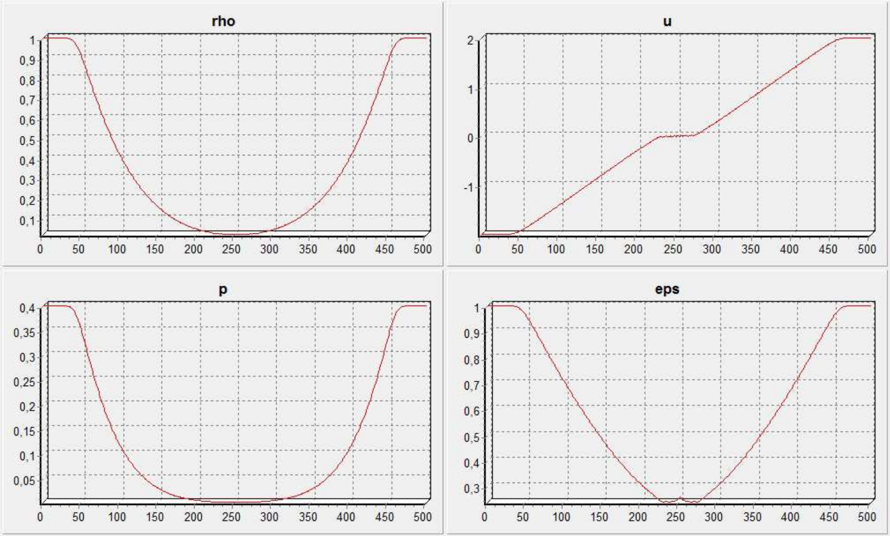


Figure 7: Two opposite rarefaction waves (results for the new scheme, N=500)

Test 3: The Noh problem

The flow is formed by the colliding of two hypersonic flows of a cold dense gas. As a result, two diverging "infinitely strong" shock waves are formed between which there remains a stationary gas with a constant density and pressure. Indeed, according to the initial conditions, the speed of sound against the unperturbed background is $c = 0.0013$. The velocity of the wave propagation is 1; i.e., the Mach number is large $M = \frac{uL}{c} = 775$. (Actually the maximum Mach number reached in the terrestrial conditions is around 30.)

Here the computational domain is $[-0.5, 0.5]$, $\gamma = 5/3$ and the initial data are

$$\rho_0(x) = 1,\ u_0(x) = \begin{cases} -1 & \text{for} \quad x \leqslant 0 \\ 1 & \text{for} \quad x > 0 \end{cases},\ p_0(x) = 10^{-6}.$$

We take $t_{fin} = 1$ and $N = 500$, $\alpha = 0.4$, $\beta = 0.03$. This β is 30 better, i.e. the time step is $30(!)$ larger, than for the standard scheme.

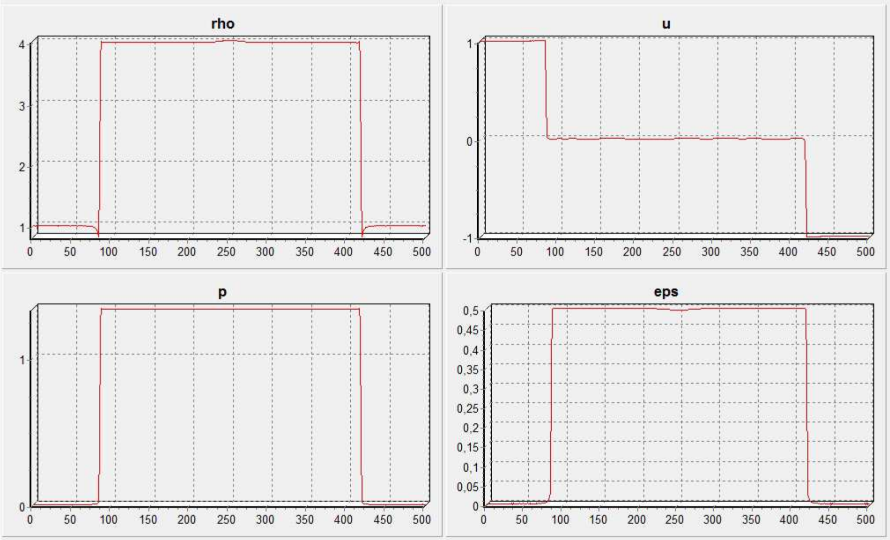


Figure 8: The Noh problem

The related publications:

1. A.A. Zlotnik and B.N. Chetverushkin, Parabolicity of the quasi-gasdynamic system of equations, its hyperbolic second-order modification, and the stability of small perturbations for them, *Comput. Math. Math. Phys.* 48 (3), 420–446 (2008).

2. A.A. Zlotnik, Energy equalities and estimates for barotropic quasi-gasdynamic and quasi-hydrodynamic systems of equations, *Comput. Math. Math. Phys.* 50 (2), 310–321 (2010).

3. A.A. Zlotnik, On construction of quasi-gas-dynamic systems of equations and the barotropic system with a potential body force, *Mat. Model.* 24 (4), 65–79 (2012) [in Russian].

4. A.A. Zlotnik, Spatial discretization of one-dimensional quasi-gasdynamic systems of equations and the entropy and energy balance equations, *Dokl. Math.* 86 (1), 464–468 (2012).

5. A.A. Zlotnik, Spatial discretization of the one-dimensional barotropic quasi-gasdynamic system of equations and the energy balance equation, *Mat. Model*. 24 (10), 51–64 (2012) [in Russian].

6. A.A. Zlotnik, Spatial discretization of one-dimensional quasi-gasdynamic systems of equations and the entropy and energy balance equations, *Dokl. Math.* 86 (1), 464–468 (2012).

7. A.A. Zlotnik, Spatial discretization of the one-dimensional quasi-gasdynamic system of equations and the entropy balance equation, *Comput. Math. Math. Phys.* 52 (7), 1060–1071 (2012).

8. V.A. Gavrilin and A.A. Zlotnik, On spatial discretization of the one-dimensional quasi-gasdynamic system of equations with general equations of state and entropy balance, *Comput. Math. Math. Phys.* 55 (2), 264–281 (2015).

9. A. Zlotnik and V. Gavrilin, On a conservative finite-difference method for 1D shallow water flows based on regularized equations, In: *Mathematical Problems in Meteorological Modelling, Math. in Industr*y, 22, 3-18. (Springer, Berlin, 2016)

10. A.A. Zlotnik, On conservative spatial discretizations of the barotropic quasi-gasdynamic system of equations with a potential body force, *Comput. Math. Math. Phys.* 56 (2), 303–319 (2016).

11. A.A. Zlotnik. Entropy-conservative spatial discretization of the multidimensional quasi-gasdynamic system of equations, *Comput. Math. Math. Phys.* 57 (4), 706-725 (2017).