Stochastic Mean-field games and control

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Stochastic Control

- **Part 1: Survey of Stochastic control**
- Brownian motion and diffusion theory
- Stochastic control problems
- Dynamic Programming and Pontriagin maximum principle
- **Part 2: Mean-field games and control**
- Existence of optimal controls
- The maximum principle

Brownian motion and Diffusion theory

The Brownian Motion

Robert Brown (1773-1858) a Scotish botanist in 1828, noticed that pollen grains suspended in water perform an irregular motion. This motion was called "Brownian motion". Brown couldn't understand what was causing it.

Brownian motion and Diffusion theory

- **Albert Einstein** (1879-1955) realised that the erratic movement, seen in Brownian motion, **was due to collisions with the molecules of water**. **The pollen grains were visible but the water molecules were not**.
- **M. Smoluchowski** (1872-1917) produced in 1906, the mathematical equations that described the Random Processes in Brownian Motion.

- **Jean Perrin** (1870-1942) then used Einstein's predictions to work out the size of atoms and remove any remaining doubts about their existence and made experiences to compute approximately the Avogadro number NA $= 6{,}022\ 140\ 857(74)\times 10^{23}$

Brownian motion and Diffusion theory

Louis Bachelier (1870-1946) in 1901, proposed a first mathematical model of the Brownian motion and applied it in finance. In particular he noticed the **Markovian nature** of the motion and linked it to the **heat equation**.
In particular he proposed the equation governing the price of a risky asset by

$$dS_t = a_t dt + b_t dW_t$$

Brownian motion and Diffusion theory

Norbert Wiener (1894-1964) in 1923 proved the existence of a Brownian motion by constructing a probability measure on the space of continuous functions(called the **Wiener measure**)

Definition. The Wiener process (Brownian motion) W_t is a stochastic process s.t:

1) $W_0 = 0$

2) (W_t) is almost surely continuous

3) (W_t) has independent increments and $W_t - W_s$ has Gaussian distribution $\mathcal{N}(0; t - s)$.

Brownian motion and Diffusion theory

Paul Lévy characterizes the Brownian motion in terms of martingales
Theorem. A continuous process $(X^1, X^2, ..., X^d)$ is a Brownian motion if and only if
i) X is a martingale with respect to P (and its own natural filtration)
ii) for all $1 \leq i, j \leq n$, $X^i(t)X^j(t) - \delta^{ij}t$ is a martingale with respect to P (and its own natural filtration).

Brownian motion and Diffusion theory

Diffusion processes and SDEs

Kolmogorov (1931). A diffusion process $(x^1, x^2, ..., x^d)$ is a Markov process with continuous paths such that:

$$i)\ E\left[(x_{t+h} - x_t)/x_s, s \leq t\right] = b(x_t)h + o(h)$$

$$ii) E\left[(x^i_{t+h} - x^i_t - hb^i(x_t))(x^j_{t+h} - x^j_t - hb^j(x_t))/x_s, s \leq t\right]$$
$$= a^{ij}(x_t)h + o(h)$$

b is the **drift** vector and (a^{ij}) is the **diffusion** matrix.

Brownian motion and Diffusion theory

Let

$$L_t = \frac{1}{2} \sum a^{ij}(t,x) \frac{\partial^2 \varphi}{\partial x^2}(x) + \sum b^i(t,x) \frac{\partial \varphi}{\partial x}(x)$$

where $a^{ij}(t,x)$ is a non negative symmetric matrix.
Then x_t is a Markov homogeneous continuous process with independent increments with L_t as its generator.
If we denote $P(s;x;t,dy)$ the transition probability function $P(s;x;t,dy) = P(x(t) \in dy / x(s) = x)$, then

Brownian motion and Diffusion theory

$$\begin{cases} \frac{\partial}{\partial t} P(s; x; t, .) = L_t^* P(s; x; t, .) \\ \lim_{t \searrow s} P(s; x; t, .) = \delta_x(.) \end{cases}$$

and

$$\begin{cases} \frac{\partial}{\partial s} P(s; x; t, .) = -L_s P(s; x; t, .) \\ \lim_{s \nearrow t} P(s; x; t, .) = \delta_x(.) \end{cases}$$

The first PDE is called the **Kolmogorov forward equation** (or the Fokker-Planck equation). The second PDE is called the **Backward Kolmogorov equation**.

Brownian motion and Diffusion theory

For a long time diffusion theory was handled (Komogorov, Feller,...) using PDEs techniques. The major drawback of this approach is that it uses a heavy machinery from **PDEs theory** and very little probability.

- Construct a transition density function $p(t, x, y)$ satisfying the PDEs
- Use the Kolmogorov-Daniell theorem to construct a Markov process with $p(t, x, y)$ as its transition desnsity function.
- Apply the Kolmogorov continuity theorem to construct a continuous version of the process.

Brownian motion and Diffusion theory

Itô's Approach

A more probabilistic approach has been suggested by **P. Lévy** and carried out by **K. Itô.**

Return to the intuitive picture of the increment, then it will satisfy

$$x(t+h) - x(t) = b(t, x(t))h + \sigma(t, x(t))(B(t+h - B(t))$$

where $\sigma(t, x(t))$ is a square root of $a(t, x(t))$ and $B(t)$ is a Brownian motion.

In differential form we obtain

$$\begin{cases} dx(t) = b(t, x(t))dt + \sigma(t, x(t))dB(t) \\ x(0) = x \end{cases}$$

Brownian motion and Diffusion theory

or in integral form

$$x(t) = x + \int_0^t b(s, x(s))ds + \int_0^t \sigma(s, x(s))dB(s)$$

Kyosi Itô have given a mathematical meaning to the integral $\int_0^t \sigma(s, x(s))dB(s)$, called Itô stochastic integral. He proved also that if b and σ are Lipschitz continuous with linear growth, the SDE has a unique solution.

Brownian motion and Diffusion theory

The cornerstone of Stochastic Calculus:The Itô formula

1) Let $f \in C_b^2(\mathbb{R})$ and (W_t) a one dimensional BM, then

$$f(W_t) = f(0) + \int_0^t f'(W_s)dW_s + \frac{1}{2}\int_0^t f''(W_s).ds$$

2) Let $f \in C_b^2(\mathbb{R}^d)$ and (W_t) a one dimensional BM, then

$$f(W_t) = f(0) + \sum_{j=1}^d \int_0^t \frac{\partial f}{\partial x_j}(W_s)dW_s^j + \frac{1}{2}\sum_{j=1}^d \int_0^t \frac{\partial^2 f}{\partial x_j^2}(W_s).ds$$

Brownian motion and Diffusion theory

Weak and Strong solution of SDEs

Definition (**Strong solution**). If X and Y are two solutions on the same probability space with the same Brownian motion, then the pathwise uniqueness holds iff: $P(X_t = Y_t,\text{for every } t) = 1.$

Definition (**Weak uniqueness**). Assume that X and Y are two solutions constructed on different probability spaces, with different Brownian motions, then the weak uniqueness holds iff the laws of X and Y coincide.

Theorem (**Yamada-Watanabe**) Pathwise uniqueness implies weak uniqueness.

Brownian motion and Diffusion theory

- **Skorokhod** (1961) proved that if b and σ are continuous and grow at most linearly, then the SDE admits a weak solution.
- **Krylov, Zvonkin and Veretennikov...**.proved various results on the existence and uniqueness of solutions of SDEs with measurable drift and elliptic diffusion coefficient, based on a corollary of Yamada-Watanabe theorem.
  Corollary. Weak existence + Pathwise uniqueness implies strong existence and uniqueness.

Brownian motion and Diffusion theory

D. W. Strook and S.R.S. Varadhan Approach

From the first relation it is not difficult to see that:
$\varphi(x(t) - \varphi(x(s)) - \int_0^t L_s\varphi(x(s)ds$ is a martingale for every $\varphi \in C_0^\infty(\mathbb{R}^d)$

Definition. P is a solution of the martingale problem if

i) $P(x(0) = x) = 1$

ii) $\varphi(x(t) - \varphi(x(s)) - \int_0^t L_s\varphi(x(s)ds$ is a P-martingale for every $\varphi \in C_0^\infty(\mathbb{R}^d)$.

Theorem (Strook-Varadhan). If b and a are measurable bounded and a is uniformly elliptic and continuous, **then the martingale problem admits a unique solution.**

Brownian motion and Diffusion theory

Cameron Martin-Girsanov theorem Let B be a Brownian motion and let u be a **nice** process. Let

$$X_t = B_t + \int_0^t u(s)ds$$

Then (X_t) is a Brownian motion under the measure Q defined by

$$\frac{dQ}{dP} = \exp\left(\int_0^t u(s)dB_s - \frac{1}{2}\int_0^t u^2(s)ds\right)$$

Remarks

1) Weak solutions via Girsanov theorem.
2) Malliavin calculus (Bismut's approach)
3) Signal processing and filtering.

Brownian motion and Diffusion theory

Another cornerstone: The Malliavin Calculus

We know how to integrate in the Itô sense. Is there any inverse operation which could be called stochastic derivative. That is can one construct a theory where

$$"\frac{\partial}{\partial W}\int_0^t h(s)dW_s = h(t)"$$

Consider the Wiener space (Ω, F, F_t, P) where $\Omega = C(\mathbb{R}_+, \mathbb{R}^d)$ and P is the Wiener measure.
Random variables in Ω are called Wiener functionals.
Let $F : \Omega \longrightarrow \mathbb{R}$ be a measurable functional in Ω.

Brownian motion and Diffusion theory

Notice that the r.v. F is defined only $P-a.s.$
Let us write formally the directional derivative of F in some direction h

$$\lim_{\varepsilon \longrightarrow 0} \frac{F(\omega + \varepsilon \widetilde{h}) - F(\omega)}{\varepsilon} = \left\langle DF, \widetilde{h} \right\rangle$$

Brownian motion and Diffusion theory

It is clear that $F(\omega+\varepsilon\widetilde{h})$ is defined $\tau^{\widetilde{h}}(P)-a.s.$ and $F(\omega)$ is defined $P-a.s.$

It is clear that the quantity $F(\omega+\varepsilon\widetilde{h})-F(\omega)$ is defined if and only if the measures P and $\tau^{\widetilde{h}}(P)$ are equivalent i.e: charge the same negligible subsets.

What are the privileged directions for which on can define a reasonable derivative?

The answer is given by the **Girsanov theorem**.

Brownian motion and Diffusion theory

It is clear that if $\widetilde{h} = \int_0^{\cdot} h(s)ds$, with $h \in L([0,1])$, then by Girsanov theorem we know that $\tau^{\widetilde{h}}(P)$ and P are equivalent. let
$\widetilde{H} = \left\{ \int_0^{\cdot} h(s)ds; \int_0^{\cdot} h(s)ds \right\}$
Moreover if $k \in \Omega - \widetilde{H}$, then $\tau^{\widetilde{h}}(P)$ is singular with respect to P, so that $F(\omega + \varepsilon k)$ is no longer defined as F is defined only $P - a.s.$
If $\widetilde{g}$ and $\widetilde{h}$ are elements of $\widetilde{H}$: $\left\langle \widetilde{g}, \widetilde{h} \right\rangle_{\widetilde{H}} = \int_0^1 g.h.ds$
H or $\widetilde{H}$ are known as the reproducing kernel space for the Gaussian measure P, in the Banach space Ω.

Brownian motion and Diffusion theory

Then

$$\lim_{\varepsilon \longrightarrow 0} \frac{F(\omega+\varepsilon\widetilde{h})-F(\omega)}{\varepsilon} = \left.\frac{d}{d\varepsilon}F(\omega+\varepsilon\widetilde{h})\right|_{\varepsilon=0} = \sum_{i=1}^{n} \partial_i f((\omega_{t_1}),...,(\omega_{t_n})).\int_0^{t_i} h.ds = \langle DF, h\rangle$$

where $DF = \sum_{i=1}^{n} \partial_i f(W(1_{[0,t_1]}),...,W(1_{[0,t_n]})1_{[0,t_i]}$
D is well defined and is H-valued.
In particular $D\left(\int_0^1 h_s d\omega_s\right) = h.$
The operator D could be defined on $\mathcal{D}^{1,2}$ = the closure of $\mathcal{S}_2$ with respect to the norm

$$\|F\|_{1,2}^2 = E\left(|F|^2\right) + E(\|DF\|_H^2)$$

Brownian motion and Diffusion theory

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Stochastic control problems

In stochastic control, the system is described by a SDE

$$x_t = x + \int_0^t b(s, x_s, u_s)\, ds + \int_0^t \sigma(s, x_s, u_s)\, dB_s$$

on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, $(B_t, t \geq 0)$ is a $d-$dim. Brownian motion, x is the initial state and u_t stands for the control variable.
The expected cost has the form

$$J(u) = E\left[\int_0^T h(t, x_t, u_t)\, dt + g(x_T)\right]$$

Stochastic control problems

The problem is to optimize the criteria $J(u)$ on some set of admissible controls $\mathcal{U}$.

$\mathcal{U}$ is a set of adapted stochastic processes with values on some action space.

Strong formulation

The filtered probability space and the Brownian motion are given in advance. The controls are adapted to this fixed filtration.

Weak formulation

The filtered probability space and the Brownian motion are parts of the control.

Stochastic control problems

Example(Merton's portfolio selection problem)
An investor invests a proportion α of his wealth X in a risky stock of constant rate of return μ and volatility σ, and the rest of proportion $(1-\alpha)$ in a bank account of constant rate interest r.His wealth X_t is then given by

$$dX_s = X_s(r + (\mu - r)\,\alpha_s)ds + X_s \sigma \alpha_s dWs,$$

and the objective is given by the value function:

$$v(t,x) = \sup_\alpha E\left[U(X_T^{t,x})\right], \, (t,x) \in [0,T] \times \mathbb{R}_+$$

U is a utility function (concave and increasing).

Stochastic control problems

2.1) Optimal stopping

The optimal control/stopping problem is to minimize

$$J(u,\tau) = E\left[\int_0^{\tau} h(t, x_t, u_t)\, dt + g(x_\tau)\right]$$

over admissible controls and stopping times.

These are typical problems encountered when dealing with American options.

N. El Karoui, Ecole d'été de St-Flour (1981).

Stochastic control problems

An example

A person who owns an asset (house, stock, etc...) decides to sell. The price of the asset evolves as:

$$dX_t = r\,X_t\,dt + \sigma\,X_t\,dB_t.$$

Suppose that there is a transaction cost $a \succ 0$. If the person decides to sell at date t, the profit of this transaction will be

$$e^{-\rho t}\,(X_t - a)\,,$$

where $\rho \succ 0$ is the inflation factor. The problem is to find a stopping time which maximizes the expected benefit:

$$\sup_{\tau} E\left[e^{-\rho\tau}\,(X_\tau - a)\right],$$

and to compute it.

Dynamic Programming

Consider the finite horizon case, where σ does not depend on the control variable.
One can associate a new control problem

$$(E_{s,y})\begin{cases} dx_t = b(t,x_t,u_t)\,dt + \sigma(t,x_t)\,dB_t, \;\; t \geq s \\ x_s = y \end{cases}$$

$$J_{s,y}(u) = E\left[\int_s^T h(t,x_t,u_t)\,dt + g(x_T)\right]$$

Consider $\mathcal{U}_{s,y}$ the set of controls over $[s,T]$.
Denote the value function $V(s,y) = \inf_{u \in \mathcal{U}_{s,y}} J_{s,y}(u)$.

Dynamic Programming

The dynamic programming principle says that if a trajectory is optimal each time, then starting from another point one can do no better than follow the optimal trajectory.

$V(s,y) = \inf_{u \in \mathcal{U}_{s,y}} E\left[\int_s^{s'} h(t, x_t, u_t)\, dt + V(s', x(s'))\right]$, $x(s')$ is the solution of $(E_{s,y})$ at s'.

Theorem (Verification theorem)

Suppose that equation

$$\text{(H.J.B)} \begin{cases} \dfrac{\partial W}{\partial t} + \inf_{u \in U} \{L^u W + h(t,x,u)\} = 0 \\ W(T,x) = g(x) \end{cases}$$

admits a $\mathcal{C}^{1,2}$ solution and that the minimum is attained at $u^*(t,x)$ such that u^* is an admissible control. Then u^* is optimal and $W = V$.

The Pontriagin stochastic maximum principle

Facing a control problem, one seeks for necessary conditions satisfied by an optimal control.
We suppose that an optimal control u^* exists. We define u^θ a perturbation of u^*.
Formally one has to differentiate the perturbed optimal cost $J(\mathrm{u}^\theta)$.

The Pontriagin stochastic maximum principle

4.1) The maximum principle for the strong formulation
Consider the controlled stochastic differential equation

$$\text{(SDE)}\ x_t = x + \int_0^t b(s, x_s, u_s)\, ds + \int_0^t \sigma(s, x_s)\, dB_s$$
$$J(u) = E\left[\int_0^T h(t, x_t, u_t)\, dt + g(x_T)\right]$$

Suppose that b, σ, h, g are smooth

Set $H(t, x, u, p) = p.f(t, x, u) - h(t, x, u)$

The Pontriagin stochastic maximum principle

Theorem Suppose that u^* is an optimal control and x^* the solution of (SDE). Then there exists an adapted process $p(t)$ such that for almost all t

$$\max_{u\in U} H(t, x_t^*, u, p_t) = H(t, x_t^*, u_t^*, p_t) \ , \ P - a.s.$$
$$p_t = E(\overline{p}_t / F_t)$$
$$\overline{p}_t' = -g_x(x_T^*)\Phi(T,t) - \int_t^T h_x(s, x_s^*, u_s^*)\Phi(s,t)ds$$

where $\Phi(s,t)$ is the unique solution of the linear equation:

$$\begin{cases} d\Phi(s,t) = b_x(t, x_t^*, u_t^*)\Phi(s,t)\,dt + \\ \qquad\qquad + \sum_{1\leq j\leq d} \sigma_x^j(t, x_t^*)\Phi(s,t)\,dB_t^j, \ \ t \geq s \\ \Phi(t,t) = I_d \end{cases}$$

The Pontriagin stochastic maximum principle

Proposition (J.M.Bismut**)** If (F_t) is the natural filtration of the Brownian motion, then $p(t)$ satisfies the linear BSDE

$$\begin{cases} dp_t = \begin{bmatrix} -b_x(t, x_t^*, u_t^*)' p_t \; - l_x(t, x_t^*, u_t^*)' + \\ + \sum_{1 \le j \le d} \sigma_x^j(t, x_t^*)' K^j \end{bmatrix} dt \; + K_t dB_t \\ p_T = -g_x(x_T^*) \end{cases}$$

The Pontriagin stochastic maximum principle

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Mean-field games

- **Game theory** is "the study of mathematical models of conflict and cooperation between intelligent rational decision-makers.
- **Game theory** is mainly used in economics, political science, and psychology, as well as logic, computer science, biology etc...
- **Existence** of mixed-strategy equilibria in two-person zero-sum games has been proved by **John Von Neumann**, (Theory of Games and Economic Behavior, **J. Von Neuman and Oskar Morgenstern**.)

Mean-field games

Remarks

- For general games, the Nash equilibrium is a solution involving two or more players, in which each player is assumed to know the equilibrium strategies of the other players, and no player has anything to gain by changing only his or her own strategy.
- **J. F. Nash** has proved existence of a stable equilibrium in the space of mixed strategies.
- **J. F. Nash**: Nobel prize in Economics (1994) and Abel Prize in Mathematics(2015), with **21 published papers!!!!**.

Mean-field games

The N Player Game.

Consider a stochastic differential game with N players, each player controlling his own private state X_t^i

$$\begin{cases} dX_t^i = b(t, X_t^i, \frac{1}{N}\sum_{i=1}^{N}\delta_{X_t^i}, u_t^i)dt + \sigma(t, X_t^i, \frac{1}{N}\sum_{i=1}^{N}\delta_{X_t^i}, u_t^i)dW_t^i \\ X_0 = x. \end{cases}$$

$$J^i(u^i) = E\left(\int_0^T h(t, X_t^i, \frac{1}{N}\sum_{i=1}^{N}\delta_{X^i}, u^i)dt + g(X_T^i, \frac{1}{N}\sum_{i=1}^{N}\delta_{X_T^i})\right)$$

$\left(u^1, u^2, ..., u^N\right)$ is a Nash equilibrium if $\forall 1 \leq i \leq N$, $\forall v \in \mathbb{A}$,

$$J^i(u^1, u^2, ., u^i.., u^N) \leq J^i(u^1, u^2, ., v.., u^N)$$

Mean-field games

Difficulties:

- Solve the HJB (Issacs PDEs) for large games.
- Numerical computations due to the dimension of the system.

When the number of players tends to infinity, can we expect some form of **averaging**?

The response is given by the MEAN-FIELD GAME (MFG) THEORY invented by PL. Lions and J. M. Lasry

The MFG theory is to search for **approximate Nash equilibriums** in the case of small players.

Mean-field games

MFG solution can be resumed in the following steps

(i) Fix a deterministic function $\mu_t \in \mathcal{P}(\mathbb{R}^d)$

(ii) Solve the standard stochastic control problem

$$\begin{cases} dX_t = b(t, X_t, \mu_t, u_t)dt + \sigma(t, X_t^i, \mu_t, u_t)dW_t \\ \inf\limits_{a \in \mathbb{A}} E\left(\int\limits_0^T h(t, X_t, \mu_t, u)dt + g(X_T, \mu_T)\right) \end{cases}$$

(iii) Determine μ_t so that $P_{X_t} = \mu_t$.

If the fixed-point optimal control identified is in feedback form, $u_t = \alpha(t; X_t; P_{X_t})$ for some function α, then

if the players use this strategy $u_t^i = \alpha(t; X_t^i; P_{X_t})$, then $\left(u_t^1, u_t^2, ..., u_t^N\right)$ should form an approximate Nash equilibrium.

Mean-field games

The solution of this problem is resumed in a coupled system of PDEs:

$$\begin{cases} \partial_t v(t;x) + \frac{\sigma^2}{2}\Delta_x v(t;x) + H(t,x,\mu_t,\nabla_x v(t,x),\alpha(t,x,\mu_t,\nabla_x v(t,x)) = 0 \\ \partial_t \mu_t - \frac{\sigma^2}{2}\Delta_x \mu_t + div_x\left(b(t,x,\mu_t,\nabla_x v(t,x),\alpha(t,x,\mu_t,\nabla_x v(t,x))\mu_t\right) = 0 \\ v(T;.) = g(;\mu_T);\ \ \mu_{0=}\delta_{x_0} \end{cases}$$

where (1) is a HJB equation and (2) is a Fokker-Planck PDE.

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Mean-field control

Optimal control of a mean-field SDE (MFSDE)

$$\begin{cases} dX_t = b(t, X_t, E(\Psi(X_t)), u_t)dt + \sigma(t, X_t, E(\Phi(X_t)), u_t)dW_t \\ X_0 = x. \end{cases}$$

The coefficients depend on the state X_t and on its distribution via a quantity $E(\Psi(X_t))$, called: **mean-field term**. Minimize

$$J(u) = E\left(\int_0^T h(t, X_t, E(\varphi(X_t)), u_t)dt + g(X_T, E(\lambda(X_T))\right)$$

over a set of admissible controls $\mathcal{U}_{ad}$.
A control $\widehat{u}$ is optimal if $J(\widehat{u}) = \inf\{J(u); u \in \mathcal{U}_{ad}\}$.

Mean-field control

The state equation (**MFSDE**) is obtained as a limit of systems of interacting particles.

Lemma

Let $(X_t^{i,n})$, $i = 1, ..., n$, *defined by*

$$dX_t^{i,n} = b(t, X_t^{i,n}, \frac{1}{n}\sum_{i=1}^{n}\psi(X_t^{i,n}), u_t)dt + \sigma(t, X_t^{i,n}, \frac{1}{n}\sum_{i=1}^{n}\Phi(X_t^{i,n}), u_t)dW_t^i$$

Then $\lim_{n\to+\infty} E\left(\left|X_t^{i,n} - X_t^i\right|^2\right) = 0$, *where* (X_t^i) *are independent and solutions of the same MFSDE.*

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Mean-field control

Applications to various fields:

- Allocation of economic resources.
- Exploitation of exhaustible resources, such as oil.
- Finance with small investors.
- Movement of large populations.

Mean-field control

Example

"**Mean-Variance Portfolio Selection**"

Consider a financial market : S_t^1(risky asset) and a bond S_t^0 (bank account) :

$$\begin{cases} dS_t^0 = \rho_t S_t^0 dt \\ dS_t^1 = \alpha_t S_t^1 dt + \sigma_t S_t^1 dW_t \end{cases}$$

If u_t is the proportion invested in S_t^1, then the value of the portfolio satisfies :

$dX_t = (\rho_t X_t + (\alpha_t - \rho_t) u_t) dt + \sigma_t u_t dW_t, X_0 = x$

Minimize the cost functional:

$J(u) = \frac{\gamma}{2} Var(X_T) - E(X_T) = \frac{\gamma}{2} \left(E(X_T^2) - E(X_T)^2 \right) - E(X_T).$

Existence of optimal controls

It is well known that an optimal control in $\mathcal{U}_{ad}$ does not exist necessarily.
Relaxed controls

- Let $\mathbb{V}$ the space of product measures $\mu_t(da)dt$ on $\mathbb{A} \times [0, T]$. Endowed with the topology of weak convergence, $\mathbb{V}$ est compact.
- A relaxed control is a $\mathcal{P}(\mathbb{A})$-valued process, which could be identified with a random variable with values in $\mathbb{V}$ written $\mu(\omega, t, da)dt$ or $\mu_t(da)dt$. Let $\mathcal{R}$ be the space of relaxed controls.

Remarque: An admissible (strict) control $u \in \mathcal{U}_{ad}$ may be identified as a relaxed control $dt.\delta_{u_t}(da)$.

Existence of optimal controls

The state equation

$$\begin{cases} dX_t = b(t, X_t, E(\Psi(X_t)), u_t)dt + \sigma(t, X_t, E(\Phi(X_t), u_t)dW_t \\ X_0 = x. \end{cases}$$

and the cost functional:

$$J(u) = E\left[\int_0^T h(t, X_t, E(\varphi(X_t)), u_t)dt + g(X_T, E(\lambda(X_T)))\right]$$

Existence of optimal controls

The relaxed state equation is given by:

$$\begin{cases} dX_t = \int_{\mathbb{A}} b(t, X_t, E\left(\Psi\left(X_t\right)\right) a)\, \mu_t(da)dt \\ + \int_{\mathbb{A}} \sigma(t, X_t, E(\Phi(X_t), a)\, M(da, dt), \\ X_0 = x \end{cases}$$

where $M(dt, da)$ is continuous orthogonal martingale measure with intensity $\mu_t(da)dt$. El Karoui et Méléard [4].
The relaxed cost is given:

$$J(\mu) = E\left(\int_0^T \int_A h(t, X_t, E(\varphi(X_t), a)\mu_t(da)dt + g(X_T, E(\lambda(X_T))\right)$$

N. El Karoui, S. Méléard, *Martingale measures and stochastic calculus*, Probab. Th. and Rel. Fields **84** (1990), no. 1, 83–101.

Existence of optimal controls

Theorem

Under $(\mathbf{H_1})$ *et* $(\mathbf{H_2})$*, an optimal relaxed control exists.*

Steps of the proof

- Let $(\mu^n)_{n\geq 0}$ a minimizing sequence, $\lim\limits_{n\to\infty} J(\mu^n) = \inf\limits_{q\in\mathcal{R}} J(\mu)$ and let X^n the state associated to μ^n.
- We prove that (μ^n, M^n, X^n) is tight.
- Using Skorokhod theorem, there exist a subsequence which converges strongly to $(\widehat{\mu}, \widehat{M}, \widehat{X})$, satisfying the state equation.
- Prove that $(J(\mu^n))_n$ converges to $J(\widehat{\mu}) = \inf\limits_{\mu\in\mathcal{R}} J(\mu)$ and conclude that $(\widehat{\mu}, \widehat{M}, \widehat{X})$ is optimal.

Existence of optimal controls

Corollary

Assume that
$P(t, X_t) = \left\{ \left(\widetilde{b}(t, X_t, E(\Psi(X_t), a) \right) ; \ a \in \mathbb{A} \right\} \subset \mathbb{R}^{d+d^2+1}$
is closed and convex, $\widetilde{b} = (b, \sigma\sigma^*, h)$. *Then the optimal relaxed control is realized as a strict control.*

The stochastic maximum principle

The state equation

$$\begin{cases} dX_t = \int_A b(t, X_t, E(X_t), a)\mu_t(da)dt + \int_A \sigma(t, X_t, E(X_t), a)M(dt, da) \\ X_0 = x, \end{cases}$$

The cost functional

$$J(\mu) = E\left[\int_0^T \int_A h(t, X_t, E(X_t), a)\mu_t(da)dt + g(X_T, E(X_T))\right].$$

The stochastic maximum principle

An optimal relaxed control exists.We derive necessary conditions for optimality in the form of Pontriagin maximum principle.
Let μ be an optimal relaxed control and X the optimal state.
The necessary conditions are given by

- two adjoint processes,
- a variational inequality.

The stochastic maximum principle

Assume

$(\mathbf{H_1})$

$$b : [0, T] \times \mathbb{R} \times \mathbb{R} \times A \longrightarrow \mathbb{R}$$
$$\sigma : [0, T] \times \mathbb{R} \times \mathbb{R} \times A \longrightarrow \mathbb{R}$$

are bounded, continuous, such that $b(t, ., ., a)$ and $\sigma(t, ., ., a)$ are C^2 in (x, y). Assume that the derivatives of order 1 and 2 are bounded continuous in (x, y, a).

$(\mathbf{H_2})$

$h : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{A} \longrightarrow \mathbb{R}$

$g : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$

satisfy the same hypothesis as b and σ.

The stochastic maximum principle

Define the first and second order adjoint processes :

$$\begin{cases} dp(t) = & -\left[\overline{b}_x(t)p(t) + E\left(\overline{b}_y(t)p(t)\right) + \overline{\sigma}_x(t)q(t) + E\left(\overline{\sigma}_y(t)q(t)\right) \right. \\ & \left. -\overline{h}_x(t) - E\left(\overline{h}_y(t)\right)\right] dt + q(t)dW_t + dM_t \\ p(T) = & -\overline{g}_x(T) - E\left(\overline{g}_y(T)\right) \end{cases}$$

$$\begin{cases} dP(t) = & -\left[2\overline{b}_x(t)P(t) + \overline{\sigma}_x^2(t)P(t) + 2\overline{\sigma}_x(t)Q(t) + \overline{H}_{xx}(t)\right]dt \\ & +Q(t)dW_t + dN_t \\ P(T) = & -\overline{g}_{xx}(x(T)) \end{cases}$$

$\overline{f}(t) = f(t, X(t), \mu(t)) = \int_A f(t, X(t), a)\mu(t, da)$ and f stands for b_x, σ_x, h_x, b_y, σ_y, h_y, H_{xx}

The stochastic maximum principle

Denote the generalized Hamiltonian

$$\mathcal{H}^{(X(.),\mu(.))}(t, Y, E(Y), a) = \\ H(t, Y, E\left(Y\right), a, p(t), q(t) - P(t).\sigma(t, X_t, E(X_t), \mu(t))) \\ -\frac{1}{2}\sigma^2(t, Y, E\left(Y\right), a)P(t)$$

where

$$H(t, X, E\left(X\right), a, p(t), q(t)) = \\ b(t, X, E(X), u).p + \sigma(t, X, E(X), u).q - h(t, X, E(X), u)$$

is the usual Hamiltonian.

The stochastic maximum principle

Theorem

(The relaxed maximum principle)

Let (μ, X) an optimal couple, then there exist (p, q) et (P, Q), solutions of adjoint equations s.t

$$E\int_0^T \mathcal{H}^{(X(t),\mu(t))}(t, X(t), \mu(t))dt = \sup_{a\in A} E\int_0^T \mathcal{H}^{(X(t),\mu(t))}(t, X(t), a)dt$$

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Thank you very much