Control and identification problems for distributed systems with persistent memory

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The equation

We present certain problems related to the controllability of the following **(GP)** equation (after Gurtin and Pipkin (1966))

$$w' = \int_0^t N(t-s)\Delta w(s)\,\mathrm{d}s\,, \quad \begin{cases} w(x,0) = w_0(x) \in L^2(\Omega)\,, \\ w(x,t) = f(x,t) \text{ if } x \in \Gamma \subseteq \Omega\,, \\ w(x,t) = 0 \text{ if } x \in \partial\Omega \setminus \Gamma \end{cases}$$

where $w = w(x,t)$ with $t > 0$ and $x \in \Omega$, a region with C^2 boundary, $N \in C^3$ and $\boxed{N(0) > 0}$.

$N(t)$ is the "relaxation kernel".

Applications of this equation

Thermodynamics for materials with memory (w is the temperature), viscoelasticity (w is the displacement) and nonfickian diffusion (i.e. in the presence of complex molecular structure: diffusion in polymers, absorption of drugs throughout the skin...) (w is the density).

For definiteness we call w the temperature but note that the equation is a linearized version of a nonlinear process. So, w is the perturbation of the temperature, not the absolute temperature.

We use the control f to control the temperature $w(T)$ at a certain time T.

(Note: in fact, under suitable conditions we can even control the pair $(w(T), w'(T))$ but here we ignore $w'(T)$)

Comments

In order to comply with the second principle of thermodynamics, the kernel must satisfy certain "positivity" and monotonicity conditions (Day, Fabrizio, Amendola, Gentili, Giorgi. . .) which are not used in the study of controllability. We only use $N(t)$ smooth on $[0, +\infty)$ and $N(0) > 0$ (for simplicity we normalize the time so to have $N(0) = 1$).

While use of **(GP)** in viscoelasticity is not debated, whether it can be accepted to model thermodynamic processes is an issue which is still much debated in physics journals. In particular negative (perturbations of the) temperature may appear when solving **(GP)**, a difficulty which is solved by passing to the second order approximation of the nonlinear problem (Fabrizio, Colemann, Owens).

The solutions

It turns out that there exists a unique solution $w \in C([0,T];L^2(\Omega)) \cap C^1([0,T];H^{-1}(\Omega))$ which depends continuously on

$$(w_0, f) \in L^2(\Omega) \times L^2(0,T;L^2(\partial\Omega))\,.$$

If $f = 0$, $w_0 \in H_0^1(\Omega)$ then

$$w \in C([0,T];H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega))$$

and depends continuously on

$$w_0 \in H_0^1(\Omega)\,.$$

Comparison with the wave/heat equation

First we note that the equation can be written as

$$w'' = \Delta w + \int_0^t M(t-s)\Delta w(s)\,\mathrm{d}s\,, \qquad M(t) = N'(t)$$

$((w(0), w'(0)) \in L^2(\Omega) \times H^{-1}(\Omega))$.

Special cases

$N(t) \equiv 1$ gives the wave equation

$$w'' = \Delta w\,;$$

$N(t) = e^{-t/\tau}$ gives the telegrapher's equation

$$w'' + \frac{1}{\tau}w' = \Delta w$$

$(1/\tau$ =Relaxation time).

If $N(t) = N_k(t) = ke^{-kt}$ then $N_k(t) \rightharpoonup \delta(t)$ and the equation with memory approximates the heat equation.

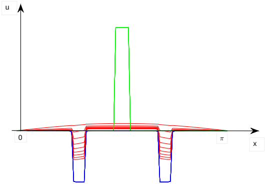
Comparison of (GP), wave and heat equations

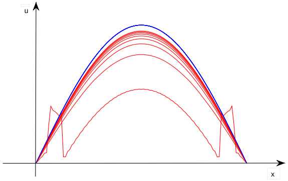
We solve the wave, **(GP)** and heat equations on $(0,\pi)$ with zero boundary conditions and initial datum (green in the left plot)

$$\xi(x) = \begin{cases} 1 & \text{if} \quad \frac{\pi}{2} - 0.1 < x < \frac{\pi}{2} + 0.1 \\ 0 & \quad \text{otherwise.} \end{cases}$$

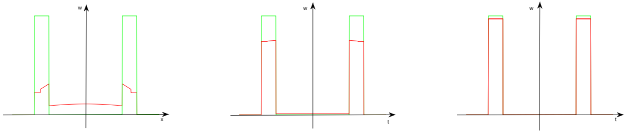
The left plot compare the solution at time $T = 2.5$ of the wave equation (blue) (zero initial velocity) and of the **(GP)** equation (red) when $N(t) = e^{-kt}$ $k = 4, 4/2, \ldots, 4/9$.

The plot on the right compare the solution of the heat equation (blue) with those of **(GP)** when $N(t) = ke^{-kt}$ $k = 4,\ 8,\ 12, \ldots,\ 36$.





Solution in free space-no boundary of the domain, $N(t) = e^{-kt}$



The solution of **(GP)** has a jump (a "forward wavefront", in the direction of the wave) as the wave equation. It does not have a wave front "backward" as the wave equation does.

The heat equation does not have wavefronts at all.

The presence of the wavefront suggests that the control properties of **(GP)** cannot mimic those of the heat equation. We conjecture that the controllability properties of **(GP)** mimic those of the wave equation.

The result

Theorem: *Let the wave equation be controllable in time T (i.e. $u^f(T)$ —solution of the wave equation controlled by f— can reach any $\xi \in L^2(\Omega)$ for a suitable f) and let $\epsilon > 0$. For every $\xi \in L^2(\Omega)$ there exists f such that* $\boxed{w^f(T+\epsilon) = \xi}$ *(w^f solves* **(GP)** *,* controlled by f).

Different proofs, in more or less general cases:

- Fourier expansion of the solution (Leugering 1984)
- observation inequality and HUM method (Kim 1993)
- Carleman estimates (interior controls) (Zhang and coworkers 2005)
- operator and moment methods (L.P. and coworkers ≥ 2004)

Here we sketch applications of operator-moment methods to control, source identification and the identification of the relaxation kernel.

Expansion in eigenfunctions

Let $\{\phi_n\}$ be an orthonormal basis of $L^2(\Omega)$ of eigenvectors of the Laplacian with homogeneous Dirichlet boundary condition ($-\lambda_n^2$ is the eigenvalue of ϕ_n). Then we have

$$w(x,t) = w^f(x,t) = -\sum_{n=1}^{+\infty} \phi_n(x) w_n(t)$$

where (γ_1 exterior normal derivative)

$$w_n(t) = \int_\Gamma \int_0^t (\gamma_1 \phi_n) \left[\int_0^s N(s-\tau) z_n(\tau)\, \mathrm{d}\tau \right] f(x, t-s)\, \mathrm{d}s\ \mathrm{d}\Gamma\,.$$

The function $z_n(t)$ solves

$$z_n{}' = -\lambda_n^2 \int_0^t N(t-s) z_n(s)\, \mathrm{d}s\,, \qquad z_n(0) = 1\,.$$

Moment problem and controllability

It is possible to hit any $L^2(\Omega)$ target if and only if the following Moment Problem is solvable for every complex sequence $\{c_n\} \in l^2$:

$$\int_0^T \int_\Gamma \left(\frac{\gamma_1 \phi_n}{\lambda_n} \right) \left[\lambda_n \int_0^s N(s-\tau) z_n(\tau)\, \mathrm{d}\tau \right] f(x, T-s)\, \mathrm{d}\Gamma\, \mathrm{d}s = c_n$$

Fact: the moment operator i.e. the transformation

$$f \mapsto \mathbb{M} f = \left\{ \int_0^T \int_\Gamma \left(\frac{\gamma_1 \phi_n}{\lambda_n} \right) \left[\lambda_n \int_0^s N(s-\tau) z_n(\tau)\, \mathrm{d}\tau \right] f(x, T-s)\, \mathrm{d}\Gamma\, \mathrm{d}s \right\}$$

is continuous from $L^2(0, T; L^2(\Gamma))$ to l^2.

Controllability and Riesz sequences

Continuity of $\mathbb{M}$ implies that $\mathbb{M}$ is surjective (i.e. controllability holds) if and only if

$$\left\{ \left(\frac{\gamma_1 \phi_n}{\lambda_n} \right) \left[\lambda_n \int_0^s N(s-\tau) z_n(\tau)\, \mathrm{d}\tau \right] \right\}$$

is a a Riesz sequence in $L^2(0, T; L^2(\Gamma))$.

A sequence $\{e_n\}$ in a Hilbert space H is a Riesz sequence when it can be transformed to an orthonormal basis of a (possibly different) Hilbert space using a linear, bounded and boundedly invertible transformation.

The case of the wave equation

When $\boxed{N(t)\equiv 1}$ our equation is the wave equation

$$u''=\Delta u\,,\quad u=f\ \mathrm{on}\,\Gamma\,,\quad u=0\ \mathrm{on}\,\partial\Omega\setminus\Gamma$$

(and zero initial conditions). It is known that the wave equation is controllable for T, Γ large enough. In the case of the wave equation,

$$\left(\frac{\gamma_1\phi_n}{\lambda_n}\right)\left[\lambda_n\int_0^s N(s-\tau)z_n(\tau)\,\mathrm{d}\tau\right]=\frac{\gamma_1\phi_n(x)}{\lambda_n}\sin\lambda_n s$$

and so for suitable T and Γ

$$\left\{\frac{\gamma_1\phi_n(x)}{\lambda_n}\sin\lambda_n s\right\}\qquad \text{is a Riesz sequence in } L^2(0,T;L^2(\Gamma)).$$

The proof of controllability-1

The proof of controllability is quite technical and we skip it. The idea is as follows.
Let $w^f(t)$ and $u^f(t)$ be the solutions of the **(GP)** and of the wave equation with the same initial condition (say equal zero) and the same control f. Let T be a control time of the wave equation, i.e.

$$\{u^f(T)\ f \in L^2(0,T;L^2(\Gamma))\} = L^2(\Omega).$$

Then the map

$$f \mapsto w^f(T) - u^f(T)$$

is compact and so

$$\mathcal{R}_T = \{w^f(T)\ f \in L^2(0,T;L^2(\Gamma))\} \subseteq L^2(\Omega)$$

is closed and has finite codimension.

The proof of controllability-2

In order to prove controllability, we use the properties of Riesz sequences to prove that

$$[\mathcal{R}_T]^\perp = 0\,.$$

This is achieved by comparing the sequences $\{e_n\}$ and $\{\epsilon_n\}$:

$$\left\{\frac{\gamma_1\phi_n(x)}{\lambda_n}\sin\lambda_n s\right\}=\{e_n\} \quad \left\{\left(\frac{\gamma_1\phi_n}{\lambda_n}\right)\left[\lambda_n\int_0^s N(s-\tau)z_n(\tau)\,\mathrm{d}\tau\right]\right\}=\{\epsilon_n\}$$

and using the fact that $\{e_n\}$ is a Riesz sequence, because the wave equation is controllable.

Comments

- It is possible to to prove controllability of the pair $(w(2T+\epsilon), w'(2T+\epsilon))$ (displacement,velocity) (in time $2T+\epsilon$)
- the arguments can be extended to different models in viscoelasticity (three-dimensional viscoelasticity, viscoelastic plates)
- when $\dim\Omega = 1$:
  - It is possible to prove controllability of the pair $(w(2T+\epsilon), q(2T+\epsilon))$ (temperature,flux)
  - It is possible to prove controllability of the pair $(w'(2T+\epsilon), \sigma(2T+\epsilon))$ (velocity of displacement,traction) and to study the control properties of $(w(2T+\epsilon), \sigma(2T+\epsilon))$ (deformation,traction).

We sum up: important sequences

This arguments show a crucial role of the sequences

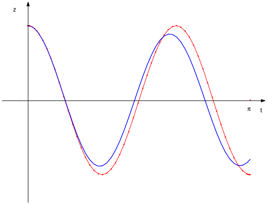
$$\left\{\left(\frac{\gamma_1\phi_n}{\lambda_n}\right)\left[\lambda_n\int_0^s N(s-\tau)z_n(\tau)\,\mathrm{d}\tau\right]\right\}\quad\text{and}\quad\left\{\frac{\gamma_1\phi_n(x)}{\lambda_n}z_n\right\}.$$

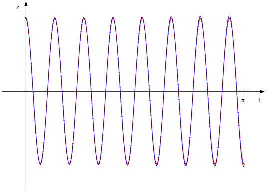
We are going to see the importance of these sequences in the solution of certain inverse problems.

- source identification
- Identification of the relaxation kernel.

How the sequence $\{z_n\}$ looks

As a preliminary information, it may be of interest to compare $z_n(t)$ with $\cos nt$ for small and large n (we use $N(t) = (1/10)e^{-t/2} + (1/5)e^{-2t} + (1/2)e^{-3t}$):





Left: low frequences; Right high frequences

- A fact that has been known from almost two centuries: at high frequences a viscoelastic material behaves very much like an elastic material.

Lord Kelvin used this property of viscoelastic materials as a model of the ether.

Source reconstruction

The problem of the source reconstruction is as follows: a source of heat, pollutant, drug,... adds energy, solute,... inside a body.

It is required to locate the position of such source using boundary measures, usually of the flux (of heat, pollutant, drug,...)

This problem has been studied using several methods. We prove that a method based on control ideas (first proposed for the wave equation (Puel, Yamamoto, Grasselli)) can be extended to the **(GP)** equation.

The model of the internal source

$$w' = \int_0^t N(t-s)\Delta w(s)\,\mathrm{d}s + b(x)\sigma(t)\,,$$
$$w(0) = 0\,,\quad w(x,t) = 0 \text{ on } \partial\Omega\,.$$

$b(x)$ is the source to be identified.

Often, $b(x)$ is the characteristic function of a subregion of Ω and $\sigma(t)$ is constant (after a transient). The method can be used if we know $\sigma(t)$ differentiable with $\sigma(0) \neq 0$ and $b \in L^2(\Omega)$. In fact, it is sufficient to know a multiple of $\sigma(t)$ to identify a multiple of $b(x)$ and this is all we need to locate the source.

Note that $b(x)$ is not constant when the "intensity" of the source is not uniform.

The observation

We observe the flux on a part of the boundary, i.e.

$$\int_0^t N(t-s)\gamma_1 w(x,s)\,\mathrm{d}s\,,\qquad x\in\Gamma\subseteq\partial\Omega\,.$$

At the expenses of a step of numerical differentiation, from this measure we can reconstruct

$$y(x,t)=\gamma_1 w(x,t)\,,\qquad x\in\partial\Omega\,.$$

We show that the Fourier coefficients of $b(x)$ can be reconstructed from y, if observed for a sufficiently long time.

The space of the observation?

We may wonder in which space the observation $y = y(x,t)$ lives. It is possible to prove that

$$y \in L^2(0,T;L^2(\Gamma))$$

(and it is a continuous function of $b(x)\sigma(t) \in L^1(0,T;L^2(\Omega))$).

An example

Let us see a plot of the output $y(t)$ when

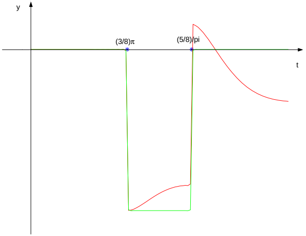
- $\Omega = (0, \pi)$
- the kernel is $N(t) = 3e^{-t} - 3e^{-2t} + e^{-3t}$
- $\sigma(t) = \mathrm{const}$ (we choose $\sigma(t) \equiv 1$)
- $b(x)$ is the characteristic function of $(\,(3/8)\pi, (5/8)\pi)$.

compared with the corresponding output of the string equation.

the flux of the wave equation and of the heat equation with memory

We compare the outputs of **(GP)** (**red**) with that of the wave equation (**green**)

$$u_t = \int_0^t u_{xx}(s)\,\mathrm{d}s + b(x) \quad (\text{we recall, } \sigma(t) \equiv 1).$$



Observation

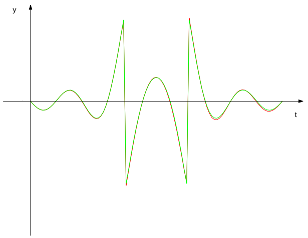
We note that (in this simple case) the output of the string equation for $t \in (0, \pi)$ precisely reproduces the "source" $b(x)$ while the output of **(GP)** is similar the output of the wave equation, but distorted.

Source identification is possible if we can "undo" the distorsion.

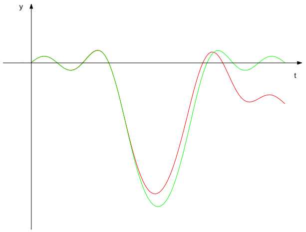
It has an interest to see the reason behind the distorsion. Let us separate the contribution of the low and high frequences both to the string and **(GP)** equation. We see that the contribution of the high frequences almost coincide while the distorsion is due to the low frequences.

Low and high frequences separated

Left: High frequences



right: low frequencies



The reason: $z_n(t) \sim \cos nt$ if $n >> 1$: a second instance of the fact that a viscoelastic material reacts to high frequency disturbances like an elastic material, while the response is different at low frequences!

Fourier expansion

As above, let ϕ_n be an orthonormal basis of $L^2(\Omega)$ with $\phi_n = 0$ on $\partial\Omega$ and $\Delta\phi_n = -\lambda_n^2\phi_n$.
Let

$$b_n = \int_\Omega \phi_n(x) b(x) \,\mathrm{d}x$$

and, as above

$$z_n' = -\lambda_n^2 \int_0^t N(t-s) z_n(s) \,\mathrm{d}s\,, \qquad z_n(0) = 1\,.$$

Accumulate the information on y

It is easily computed

$$w(x,t)=\sum_{n=1}^{+\infty}\phi_n(x)\left[b_n\int_0^t z_n(t-s)\sigma(s)\,\mathrm{d}s\right]$$

$$y(x,t)=\sum_{n=1}^{+\infty}(\gamma_1\phi_n)\left[b_n\int_0^t z_n(t-s)\sigma(s)\,\mathrm{d}s\right]$$

Fix an interval $[0,T]$ and accumulate the information in y by computing integrals

$$\int_\Gamma\int_0^T h(T-s)y(x,s)\,\mathrm{d}s\;\mathrm{d}\Gamma$$

(function h to be determined below).

few manipulations

Few manipulations give the equality

$$\int_\Gamma \int_0^T h(x, T-s) y(x,s) \,\mathrm{d}s \;\mathrm{d}\Gamma =$$

$$= \int_\Omega b(x) \left[\sum_{n=1}^{+\infty} \phi_n \left(\int_\Gamma \int_0^T (\gamma_1 \phi_n)\, z_n(r) (h * \sigma)(T-r) \,\mathrm{d}r \;\mathrm{d}\Gamma \right) \right] \mathrm{d}x$$

($*$ denotes the convolution).

The conditions on σ

We know that σ is differentiable and $\sigma(0) \neq 0$. Then, for every $f \in L^2(0, T)$ there exists h such that

$$h * \sigma = \int_0^t \sigma(t-s)h(s)\,\mathrm{d}s = \int_0^t N(t-s)f(s)\,\mathrm{d}s = N * f\,.$$

We insert this expression of $h * \sigma$ and we find the equality

$$\int_\Gamma \int_0^T h(x, T-s) y(x,s)\, \mathrm{d}s \; \mathrm{d}\Gamma = \int_\Omega b(x) w^f(x,T)\, \mathrm{d}x$$

where $w^f(t)$ be given by

$$\begin{aligned} & w' = \int_0^t N(t-s)\Delta w(s)\, \mathrm{d}s \,, \qquad w(0) = 0 \\ & w(x,t) = f(x,t) \text{ if } x \in \Gamma \,, \\ & w(x,t) = 0 \text{ if} x \in \partial\Omega \setminus \Gamma \,, \end{aligned}$$

a controlled **(GP)**, with control f.

Controllability and identification

Thanks to controllability at time T, we can choose f (and the corresponding h) which forces the bracket to take any prescribed value in $L^2(\Omega)$ at time T. In particular, we choose $h = h_k$ such that is equal to $\phi_k(x)$. With this function h_k, we find

$$\int_0^T \int_\Gamma h_k(x, T-s) y(x,s)\, \mathrm{d}\Gamma \ \mathrm{d}t = b_k = \int_\Omega b(x)\phi_k(x)\, \mathrm{d}x$$

and so

$$b(x) = \sum_{n=1}^{+\infty} \phi_n(x) \left[\int_0^T \int_\Gamma h_k(T-s)\,(\gamma_1 w)\,(s)\, \mathrm{d}\Gamma \ \mathrm{d}t \right].$$

This is the reconstruction of the unknown source $b(x)$.

Identification of the relaxation kernel

Source identification uses that the relaxation kernel $N(t)$ is known.

The memory kernel $N(t)$ and $M(t) = N'(t)$ are material properties and only a qualitative behavior can be imposed from theoretical considerations. In practice, the function $N(t)$ has to be identified on the basis of experimental measures taken on samples of the material.

Kernel identification is a very important technical problem, much studied both in mathematical and in engineering journals without much interaction between these two worlds.

Existing algorithms

- Mathematical literature: highly nonlinear algorithms for the identification of the pair (N, w). These algorithms require the solution of a system of nonlinear integrodifferential equations in Hilbert spaces (Lorenzi, Grasselli, Kabanikin, Guidetti. . .) .
- Engineering literature: $N(t)$ is assumed to depend on "few parameters". In general $N(t)$ is a Prony sum, i.e. $N(t) = \sum \alpha_n e^{-\beta_n t}$. The "real" parameters $\alpha_n > 0$, $\beta_n \geq 0$ are chosen by minimizing the (quadratic) discrepancy from experimental measures and theoretical values of the output.

Our goal: identification of the relaxation kernel using a "linear algorithm"

"Linear algorithm"=an algorithm which inverts only linear operators.

The algorithm uses two boundary measures but it is is linear and it does not assume any special class of kernels, a part

- regularity
- $N(0) > 0$. $\sqrt{N(0)}$=velocity of propagation, easily measured. So we can assume this measure already done and $N(0)$ normalized to $N(0) = 1$. Only for simplicity of presentation.

Fact: regularity rules out Abel kernels, i.e. combinations of $N(t) = \sum \alpha_n / t^{\beta_n}$. Extension of the algorithm to this case is possible, with different proofs.

A warning

Only for simplicity we assume that we test a sample in the form of a rod (say on $(0, \pi)$).

We take two measurements:

- first measurement: flux due to a nonzero initial temperature;
- second measurement: flux due to a nonzero boundary temperature, applied to $x = 0$.

In both the cases the observed output is

$$q(t) = \int_0^t N(t-s) w_x(\pi, s) \, \mathrm{d}s = (\text{flux at } x = \pi)$$

(which is the flux with the sign changed).

The flux due to the initial condition

We impose zero boundary conditions and initial condition $\xi \neq 0$. Using the expansion in eigenfunctions we find

$$\frac{\pi}{2} q(x,t) = \sum_{n=1}^{+\infty} \xi_n \left[n \int_0^t N(t-s) z_n(s)\, \mathrm{d}s \right] \cos nx$$
$$= -\sum_{n=1}^{+\infty} \frac{1}{n} \xi_n z_n'(t) \cos nx$$

the FIRST measurement

The flux due to a special boundary condition

choose $\xi_0(x) = \frac{1}{2}(\pi - x) = \sum_{n=1}^{+\infty} \frac{1}{n} \sin nx$. Measure the flux at π:

this measurement provides the function:

$$K(t) = q(t) = -\sum_{n=1}^{+\infty} \frac{1}{n^2}(-1)^n z_n'(t)\,.$$

The boundary temperature

Impose null initial condition to a sample of the material, and boundary condition

$$w(0,t)=f(t)\,,\quad w(\pi,t)=0\,.$$

Second measurement: $Y^f(t)$ is minus the flux at π.

We assume

$$f(t)=\int_0^t g(s)\,\mathrm{d}s\,,\qquad g'(0)\neq 0\,.$$

Note: the boundary temperature in practice cannot be arbitrary. In particular it must be zero at $t=0$ (since $\xi=0$) and it must "saturate", as for example

$$f(t)=1-e^{-t}\quad\text{or}\quad f(t)=1-\frac{1}{t+1}\,.$$

the SECOND measurement

It turns out that the flux at $x = \pi$ due to the temperature

$$f(t) = \int_0^t g(s)\,\mathrm{d}s$$

is (minus) the output $Y^f(t)$ we get in the second measurement

$$\frac{\pi}{2} Y^f(t) = \int_0^t g(t-s)\left(K(s) - \frac{1}{2} N_1(s) \right)\,\mathrm{d}s$$

$N_1(t) = \int_0^s N(r)\,\mathrm{d}r$ and (miracle!!) we already measured $K(t)$.

Conclusion

$$N_1(t) = \int_0^t N(s)\,\mathrm{d}s$$

is the solution of

$$\int_0^t g(t-s)N_1(s)\,\mathrm{d}s = 2\int_0^t g(t-s)K(s)\,\mathrm{d}s - \pi Y^f(t) :$$

The right hand side is known and so the computation of $N_1(t)$ is a simple deconvolution. The computation of $N(t) = N_1'(t)$ is a numerical differentiation.

Warning

Deconvolution to compute $N_1(t)$ is simple but every algorithm (Tikonov, Lavrentev,...) amplifies the noise. So, the computation of the numerical derivative $N_1'(t) = N(t)$ is not that simple.

But, the physics of the problem suggest several remedies: $N(t)$ must be positive decreasing (often convex) so that $N_1(t)$ is positive increasing concave. Taking these properties into account we can regularize the reconstruction of $N_1(t)$ for example using averages on nearby steps.
We then use any algorith for numerical differentiation and we get $N(t)$.

A toy simulation

Let the "unknown" relaxation kernel be the Prony sum

$$N(t) = (1/10)e^{-t/2} + (1/5)e^{-2t} + (1/2)e^{-3t}$$

(kernel of this form are often encountered in the Engineering literature.)

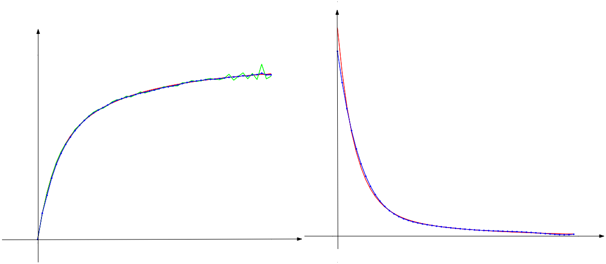
We can (approximately) compute $w(x,t)$ and the corresponding fluxes due to the initial and boundary temperatures (plus an artificial 1% error) and we can "reconstruct" $N_1(t)$ and $N(t)$ from these data.

The plots

We present the plots of the "true" kernels $N_1(t) = \int_0^t N(s)\, \mathrm{d}s$ and $N(t)$ versus their numerical reconstructions, using the algorithms described above. This requires the solution of deconvolution problems which are ill posed. So, we must relay on suitable regularization. We used Lavrentev regularization for the reconstruction of $N_1(t)$, followed by an averaging regularization (since we know from physics that $N(t)$ is increasing concave). The reconstruction of $N(t)$ from $N_1(t)$ requires a numerical differentiation which we obtained in the most elementary way, by fitting a polynomial to the reconstruction of $N_1(t)$.

The “true” and reconstructed functions

Figure: Left: $N_1(t)$ (Lavrentev algorithm green, then regularized by averaging); right: the relaxation kernel $N(t)$ (red are the “true” plots).



An apparent difficulty

In practice, there is no difficulty to impose even time varying boundary temperature while it is essentially impossible to prescribe a "strange" initial temperature. We must use a physically realizable initial temperature in order to have a practical algorithm.

Luckily, the required initial temperature ξ (plot on the left) is easily realizable by imposing two different (constant) temperatures at the ends of the bar, and waiting for the equilibrium.

A variant and a warning: the model of Colemann-Gurtin

A related but different model which is also used in fluid dynamics is

$$w_t = \Delta w + \int_0^t N(r-s)\Delta w(s)\, \mathrm{d}s\,.$$

This model has been first introduced by Jeffrey (1922) and more in general by B.D Colemann and G. Gurtin (1967).

The controllability properties of this equation are very different from those of **(GP)**.

The end

Thanks to the participants for the attention and to the organizers for the kind invitation